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ПРОФЕССИОНАЛЬНЫЙ АНГЛОЯЗЫЧНЫЙ ДИСКУРС: МАТЕМАТИКА

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Учебное пособие выполнено в рамках научной школы Института иностранных языков МГПУ «Межкультурное многоязычное образование как фактор социальных трансформаций»

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Учебное пособие по английскому языку для специальных целей предназначено для обучающихся математических направлений и профилей подготовки, а также студентов, изучающих смежные дисциплины, такие как физика и информатика, для которых математический курс является общепрофессиональной дисциплиной.

Пособие содержит обзор теории, истории и междисциплинарных связей математики, сопровождаемый разнообразными упражнениями и заданиями, что позволяет последовательно овладеть терминами и понятиями, необходимыми для осуществления профессиональной коммуникации на английском языке.

Содержание пособия способствует развитию межкультурной коммуникативной компетенции обучающихся в сфере профессионального общения (математика).

Цифровой формат пособия позволяет эффективно использовать его в рамках как аудиторной, так и внеаудиторной работы.

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Оглавление

Методические рекомендации

Настоящее учебное издание предназначено для студентов, обучающихся на математических направлениях подготовки и владеющих английским языком на уровне выпускника средней общеобразовательной школы. Пособие представляет собой практический курс английского языка для специальных целей (математика).

Целью пособия является развитие межкультурной коммуникативной компетенции обучающихся в сфере профессионального общения (математика). К учебным задачам освоения пособия относятся:

- развитие умений чтения текстов, связанных с предметной сферой *математика*;
- развитие медиативных умений сужения, адаптации и расширения вербальной информации в текстах математической тематики;
- развитие умений письменной речи, говорения и аудирования указанной предметной сферы;
- совершенствование лексических навыков в области математической терминологии.

Пособие базируется на концепциях и принципах, изложенных в трудах И. Л. Бим [Бим 1999], Е. Г. Таревой [Тарева 2014; 2017; 2018], С. В. Гринева-Гриневича [Гринев-Гриневич 2005], К. Н. Бурнаковой [Бурнакова 2018], А. А. Колесникова [Колесников 2023], Т. С. Макаровой [Макарова 2020], Л. А. Миловановой [Милованова 2023], Д. Кристала [Crystal 2003].

Структура и тематика данного пособия способствуют более глубокому ознакомлению обучающихся с иной лингвокультурой [Амиров 2018; Tareva, Tarev 2018; Чупрына 2018; 2021] и способствуют становлению «личностного тезауруса специалиста с целью эффективного функционирования в профессиональной сфере» [Абдулмянова 2010, с. 36].

Пособие предусматривает возможность выполнения ряда заданий в цифровом формате, что способствует индивидуализации процесса освоения английского языка для специальных целей, позволяя обучающемуся в определенных случаях совершенствовать языковые навыки за счет многократного повторения материала и наличия зрительных и аудио опор.

Значимым в данном пособии являются:

- нацеленность на обучение студентов неязыковых специальностей и индивидуализацию их обучения [Terra autonomia 2022; Аймалетдинов 2023; Апарина 2023; Воеводина 2023], создание адаптивной цифровой образовательной среды

[Vikulova, Khutyz, Makarova [et al.] 2020; Современная {цифровая} дидактика 2024], учет гетерогенного характера группы обучающихся [Гурова 2023];

- опора на аутентичный материал, в том числе публицистический [см. Зоидзе 2012; English as a means of communication: basic course 2023], художественный [см. Абаева 2016; Николаева 2022; Баканова 2024], научные тексты [см. Будник 2018; Абаева, Стекольщикова 2022; Данилова 2024], трейлеры и фрагменты художественных фильмов [Павлова 2025], аудиоподкасты [Дорохова 2022], эргонимы современного мегаполиса [Матюшина 2024], а также тексты из учебной и справочной литературы и медиа источников, предназначенные для носителей английского языка и адаптированные с учетом образовательных потребностей русскоязычных обучающихся высшей школы;

- тематика отобранных текстов, способствующих ознакомлению обучающихся с основными положениями лингвистической науки [см. Лягушкина 2009; Абаева, Стекольщикова 2022; Матвеева 2024], формированию межкультурной коммуникативной компетенции, основ профессиональной иноязычной культуры студентов неязыковых вузов [Черкашина 2012]; в основу отбора текстов легли положения, касающиеся учета профессионального профиля будущих математиков [Черкашина 2011];

- учет вариативности математической терминологии в британском и американском вариантах английского языка и возможности межвариантной интерференции в математическом дискурсе;

- включение коммуникативных парных и групповых заданий, нацеленных на развитие практических навыков по культуре речевого [см. Практикум по культуре 2016], а также профессионального общения [см. Практикум профессионального 2020], различных проектных заданий [Makarova, Matveeva, Molchanova [et al.] 2017; Grubin 2022] и современных методов обучения [Цыганкова 2023; Makarova, Matveeva, Molchanova [et al.] 2023];

- наличие цифровых упражнений для самостоятельной работы студентов [см. Макарова 2009; Suleimanova 2020; Гурова 2023], предназначенных для использования в электронном [см. Zoidze, Matveeva, Makarova [et al.] 2021], дистанционном [см. Dugina 2020] и гибридом форматах обучения [см. Постмодернизм 2020];

- привлечение информационно-коммуникационных технологий [Suleimanova 2020; Фролова 2022; Фононовации 2024] в том числе видео практикумов [см. Моралова 2024; Юнгина 2024].

Учебное пособие состоит из шести разделов: «Введение в математику», «Теория чисел и алгебра», «Геометрия», «Разделы высшей математики», «История математики», «Математика и смежные науки». Каждый раздел включает пять тем — всего 30 тем.

Каждая тема предусматривает последовательное совершенствование лексических навыков и развитие коммуникативных умений обучающихся в профессиональных контекстах на материале текстов математической тематики. Терминологический минимум находит отражение как в самих текстах, так и в заданиях, предтекстовых, притекстовых и послетектовых, языковых, речевых и коммуникативных.

При проектировании заданий пособия авторы опирались на мнение о том, что лингвообразовательный процесс будет «варьироваться в зависимости от профиля подготовки, исходного уровня студентов, особенностей их профессионального мышления и профессионального сознания» [Черкашина 2017, с. 309]. Поэтому в пособии представлены разные типы упражнений – от классических подстановочных до проектных заданий, в определенной степени моделирующих профессиональную деятельность.

Пособие включает задания следующих типов:

- задания на усвоение терминологии (значение, орфография, произношение, сочетаемость, словообразование);
- задания на понимание и переосмысление содержания текстов;
- речевые и коммуникативные задания с включением изучаемых терминов и терминосочетаний;
- исследовательские и творческие задания для развития умений решения профессиональных задач, а также углубления и расширения знания предметной области.

Разработка исследовательских и творческих заданий осуществлялась с учетом положений, регламентирующих методологию использования средств генеративного искусственного интеллекта в образовании [Тивьяева, Михайлова, Казанцева 2024; Тивьяева, Михайлова 2025].

Пособие может представлять интерес для студентов смежных направлений и профилей подготовки, изучающих математические дисциплины в качестве общепрофессиональных курсов (физика, информатика).

Требования к уровню освоения содержания учебного пособия

В результате освоения материала учебного пособия студент овладевает основными понятиями и терминами математики на английском языке, в том числе умениями воспринимать и понимать их на слух и зрительно, а также адекватно использовать их в своей речи в ситуациях профессионального общения. Обучающийся развивает умения анализа и синтеза научно-популярных, научных и других типов текстов по математической тематике и проблематике, приобретает представления о теории и истории математики. Ожидается, что освоение учебного материала пособия подготовит студента к самостоятельной работе со специальной литературой и общению на английском языке для успешного ведения трудовой деятельности и эффективного коммуникативного взаимодействия в профессиональном дискурсе.

Методические рекомендации студентам по работе с учебным пособием

Одним из условий успешного освоения содержания учебного пособия является чтение текстов и самостоятельная работа со словарями. В связи с этим чтение иноязычных профессионально-ориентированных текстов организовано в рамках традиционной технологии предтекстового, текстового и послетекстового этапов. Предтекстовый этап направлен на смысловую антиципацию последующей работы с текстом, семантизацию базовых терминологических единиц, выявление профессионально-коммуникативных дефицитов. На данном этапе предполагается развитие коммуникативных умений обучающихся в процессе групповой работы в форме мозговых штурмов, дебатов, ролевых игр и т. п. На текстовом этапе предполагается формирование умений поискового, ознакомительного, изучающего чтения от определения и выделения основной информации до полного и максимально точного понимания содержания текста. Иноязычный профессионально-ориентированный текст рассматривается в контексте обучения информационно-поисковой деятельности, формирования навыка читательской самостоятельности и обучения чтению как опосредованной форме профессионально-ориентированной коммуникации. Данная задача решается на послетекстовом этапе. В качестве основных заданий для предтекстового, текстового и послетекстового этапов предлагаются задания на выявление ассоциаций, задания, направленные на коллективный поиск решений, прогнозирование содержания текста, а также словарная работа, обсуждение ситуации, связанной с тематикой текста, поиск ключевых слов и выражений, ответы на вопросы, выделение главной мысли абзацев, трансформация предложений, заполнение пропусков, задания творческого и дискуссионного характера.

При работе с математическими терминами студенту рекомендуется:

1. С помощью монолингвальных и билингвальных словарей и справочников выверять каждый термин и терминосочетание — его правописание, произношение, значение, межъязыковую эквивалентность.
2. Вести индивидуальный словарь математических терминов с переводом и фиксацией типичных словосочетаний.
3. Работать над запоминанием определений путем многократного повторения, самостоятельного воспроизведения, а также с опорой на предложенные интерактивные упражнения.

Важно учесть, что ряд заданий в пособии, отмеченных символом «ноутбук», предназначен для выполнения в цифровом виде. Такие задания могут выполняться как в аудиторное время (с использованием компьютеров, планшетов или интерактивной доски), так и во внеаудиторной самостоятельной работе.

Цифровой формат позволяет автоматизировать проверку, мгновенно получать обратную связь, а также дает возможность многократного прохождения упражнений для закрепления материала.

Методические рекомендации преподавателю по использованию учебного пособия

Учебное пособие предназначено для студентов, получающих высшее образование по направлениям и профилям подготовки в области математики и смежных наук, изучающих английский язык как средство профессионального общения и владеющих им на уровне не ниже порогового.

Содержание пособия является гибким по временным затратам. Оно может быть освоено в рамках одного учебного года, либо в течение более длительного периода учебного времени в зависимости от уровня подготовки студентов, количества часов, предусмотренных учебным планом, и конкретных целей освоения дисциплины. Преподаватель вправе самостоятельно корректировать темп прохождения тем, выбирать наиболее релевантные задания и определять долю аудиторной и внеаудиторной работы.

В обучении языку для специальных целей целесообразно следовать педагогическим подходам: системному, коммуникативно-деятельностному и личностно-ориентированному, учитывать специфику этапа обучения, профиля обучения, а также уровня сформированности иноязычной коммуникативной компетенции обучающихся.

Тексты пособия посвящены актуальным для современного профессионала в области математики темам. Рекомендуется апеллировать к мотивационно-волевой сфере обучающихся, культивировать профессиональный интерес и развивать профессиональную личность обучающихся через обращение к личному, учебному и трудовому опыту обучающихся.

Пособие не претендует ни на всеобъемлющий охват вопросов математики, ни на полноту отраженной терминологии. Оно является введением в язык специальности и служит основой для дальнейшего непрерывного развития профессиональной коммуникативной компетенции.

Главным фокусом аудиторной работы является развитие умений устной профессионально-ориентированной речи, лексико-грамматическая база которой создается в том числе и за счет внеаудиторной работы с цифровым содержание учебного пособия.

При работе с цифровыми заданиями, обозначенными символом «ноутбук», рекомендуется следующая организация процесса обучения.

На аудиторных занятиях рекомендуется использовать цифровые задания на этапе закрепления материала или оперативного контроля. Студенты выполняют упражнения на персональных компьютерах, ноутбуках или мобильных устройствах, результаты могут обсуждаться в парах или группах. Преподаватель при необходимости комментирует типичные ошибки, используя проекцию экрана.

Во внеаудиторной работе следует предлагать студентам цифровые задания для самостоятельного выполнения с автоматической проверкой, например, в формате интерактивных тестов, упражнений на подстановку, соответствие, порядок слов. Это позволяет перенести рутинные виды работы в цифровую среду и освободить аудиторное время для выполнения коммуникативных задач.

Цифровые задания также можно встраивать в электронные курсы (LMS, облачные сервисы). Преподаватель может отслеживать прогресс каждого студента, анализировать наиболее частые ошибки и давать точечные конкретные рекомендации.

Цифровые задания, интегрированные в пособие, дополняют упражнения, представленные в традиционном формате, обеспечивая индивидуализацию, интерактивность и непрерывность процесса обучения.

"All is number." (Pythagoras)

Module 1. Mathematics 101

Unit 1. Digits and Numbers

Figure 1. "Pythagoreans Celebrate the Sunrise" (by Fyodor Bronnikov)

Activity 1. Add the missing forms in the chart and complete the sentences with the words.

Noun (Full)	**Noun (Abbreviated)**	**Noun (Person)**	**Adjective**	**Adverb**
mathematics	(British) (American)			

1. __________ is a fundamental subject that explores the relationships between numbers, shapes, and patterns.
2. __________ reasoning is essential for solving real-world problems and making informed decisions.
3. I enjoy solving challenging __________ problems; it's a great way to exercise my brain.
4. In my __________ class, we are learning about algebraic equations and their applications.
5. The __________ devoted his life to unraveling the mysteries of prime numbers.
6. The professor explained the concept __________, using rigorous proofs and logical deductions.

Activity 2. Fill in the blanks with the word "digit" or "number" in the correct form (singular or plural).

A symbol that forms part of a (1) __________ is called a (2) __________. For example, the (3) __________ 42.768 has five (4) __________. Ten (5) __________ are used in the decimal system, namely, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 to write any (6) __________.

The word (7) __________ also means a finger or a toe. As one learns to count with one's fingers it is not surprising that the word has come to be used for specific (8) __________ the fingers represent.

Activity 3. Match the words with the definitions.

1. a number	a. a symbol that represents a number
2. a number of	b. a sign or word that represents an amount or quantity
3. a numeral	c. to name each one of a series or list of things
4. numerical	d. existing in large numbers
5. numerous	e. basic skills in mathematics
6. numerate	f. expressed as numbers, or consisting of numbers
7. numeracy	g. having basic skills in mathematics
8. to enumerate	h. an unspecified amount, several, or many
9. enumeration	i. the act or process of naming each one of a series or list of things
10. innumerable	j. too many to be counted

Activity 4. Choose the best alternative.

1. A large number / numeral of students demonstrated proficiency in numeracy on the standardized test.
2. A numeral / numerical is a symbol representing a number.
3. Being numerate / numerous requires a solid understanding of arithmetic and mathematical principles.
4. Enumerate / Innumerable the prime numbers between 1 and 20.
5. Enumerate / Numeracy skills are crucial for interpreting statistical data in research studies.
6. Numerate / Innumerable combinations are possible when arranging the coloured tiles in the grid.
7. Numerate / Numerous factors contribute to the complexity of this mathematical problem.
8. The enumerate / enumeration of solutions to the equation revealed multiple valid answers.
9. The number / numeral of apples in the basket is 15.
10. The numerical / numerous solution to the equation is 8.

Activity 5. Study the tables.

Table 1

Cardinal Numbers							
0	zero, nought	10	ten				
1	one	11	eleven				
2	two	12	twelve	20	twenty		
3	three	13	thirteen	30	thirty		
4	four	14	fourteen	40	forty		
5	five	15	fifteen	50	fifty		
6	six	16	sixteen	60	sixty		
7	seven	17	seventeen	70	seventy		
8	eight	18	eighteen	80	eighty		
9	nine	19	nineteen	90	ninety		

Table 2

Ordinal Numbers					
0th	zeroth (zeroeth)	10th	tenth		
1st	first	11th	eleventh		
2nd	second	12th	twelfth	20th	twentieth
3rd	third	13th	thirteenth	30th	thirtieth
4th	fourth	14th	fourteenth	40th	fortieth
5th	fifth	15th	fifteenth	50th	fiftieth
6th	sixth	16th	sixteenth	60th	sixtieth
7th	seventh	17th	seventeenth	70th	seventieth
8th	eighth	18th	eighteenth	80th	eightieth
9th	ninth	19th	nineteenth	90th	ninetieth

Table 3

In writing, two-word numerals from 21 to 99 are hyphenated (-).	
twenty-one ninety-nine	twenty-first ninety-ninth

Table 4

100	a/one hundred	100th	hundredth
1,000	a/one thousand	1,000th	thousandth
1,000,000	a/one million	1,000,000th	millionth
1,000,000,000	a/one billion	1,000,000,000th	billionth
1,000,000,000,000	a/one trillion	1,000,000,000,000th	trillionth

Table 5

a single digit a single figure a one-digit number a one-figure number a single-digit number	a multidigit, a multidigit number a multifigure, a multifigure number	
	a double digit a double figure a two-digit number a two-figure number a double-digit number	a triple digit a triple figure a three-digit number a three-figure number a triple-figure number
1, 2, 3	*12, 34, 56*	*123, 456, 789*

Table 6

987 nine hundred and eighty-seven (BrE)
987 nine hundred eighty-seven (AmE)
987 nine eighty-seven (AmE, informal)

Table 7

1100 one thousand one hundred OR eleven hundred
9900 nine thousand nine hundred OR ninety-nine hundred

Table 8

In high numbers, the words "hundred", "thousand" etc. do not take an "s".	**"s" is used when referring to approximate quantities and is often followed by "of".**
24 two dozen	dozens of
300 three hundred	hundreds of
4,000 four thousand	thousands of
5,000,000 five million	millions of
6,000,000,000 six billion	billions of
7,000,000,000,000 six trillion	trillions of

Table 9

Negative Numbers **Positive Numbers**

−2	−1	0	1	2
minus two	minus one	zero	one	two
negative two	negative one	nought		

Table 10

	Whole Number Part				Decimal Point	**Fractional Part**		
Places	Thousands	Hundreds	Tens	Ones Units		Tenths	Hundredths	Thousandths
Place Values	1	2	3	4	.	5	6	7
	1,234.567 one thousand two hundred thirty-four point five six seven							

Table 11

In English, integers (whole numbers) are written with commas (,) and decimals with dots (.).
123,456 one hundred twenty-three thousand four hundred fifty-six **456.789** four hundred fifty-six POINT seven eight nine

Table 12

Fractions	
Fractions **Vulgar Fractions** **Common Fractions**	**Decimals** **Decimal Fractions**
$\frac{1}{2}$ a/one half, one over two $\frac{4}{2}$ four halves, four over two $\frac{1}{4}$ a/one quarter, a/one fourth, one over four $\frac{2}{4}$ two quarters, two fourths, two over four $5\frac{3}{7}$ five and three sevenths, five and three over seven	**0.5** point five, zero point five, nought point five **1.5** one point five **11.55** eleven point five five **123.456** one hundred twenty-three point four five six

Table 13

$\frac{numerator}{denominator}$			
a proper fraction	$\frac{3}{4}$	**an improper fraction**	$\frac{4}{3}$
a complex fraction	$\frac{3/5}{4/7}$		
a mixed fraction **a mixed number**	$3\frac{1}{2}$	**a mixed decimal** **a decimal mixed number**	2.38

Table 14

Percentages	Fractions, Decimals, Percentages
1% one per cent (percent) **10%** ten per cent **12.34%** twelve point three four per cent	$\frac{9}{10}$ nine tenths, nine over ten **0.9** (zero/nought) point nine **90%** ninety per cent

Table 15

Ratios	Proportions
1:3 one to three	**A:B = C:D** A is to B as C is to D $\frac{A}{B} = \frac{C}{D}$ A is to B as C is to D

Activity 6. Read aloud the numbers.

1) 0
2) 31st
3) 42nd
4) 53rd
5) 64th
6) 75th
7) 86th
8) 97th
9) 300
10) 400
11) 500
12) 6,000
13) 7,000
14) 8,000
15) 132
16) 132nd
17) 243
18) 243rd
19) 354
20) 354th
21) 4,765
22) 4,765th
23) 5,876
24) 5,876th
25) 6,987
26) 6,987th
27) 2,000,000,000
28) $\frac{3}{2}$
29) $\frac{1}{3}$
30) $\frac{2}{4}$
31) $\frac{3}{5}$
32) $3\frac{5}{7}$
33) $4\frac{6}{8}$
34) $5\frac{7}{9}$
35) 0.312
36) 3.1
37) 4.21
38) 55.55
39) 654.456
40) 789.9
41) 2%
42) 31%
43) 50%
44) 64.5%
45) 70.79%
46) 87.987%

Activity 7. Label each number an integer or a decimal. Read them out loud.

1) 1,946
2) 7,498
3) 34,209
4) 52,867
5) 76,804
6) 85,432
7) 296,500
8) 6.409
9) 498,903
10) 535,600
11) 124.575
12) 8.702
13) 816,492
14) 37.897
15) 26.936
16) 231.973

Activity 8. Make a list of ten multidigit numbers: cardinal, ordinal, integers, fractions, decimals, percentages. Without showing the list, read it aloud for your partner to write down symbolically. Check if the numbers are correct. Swap roles.

Activity 9. Name the place of each digit within the numbers.

1) 50
2) 501
3) 519
4) 950.1
5) 95.01
6) 9.015
7) 195.91
8) 915.159

Activity 10. Give examples to illustrate the notions.

1. a digit
2. a cardinal number
3. an ordinal number
4. a single-digit number
5. a multidigit number
6. a double figure
7. a triple figure
8. a place
9. a fraction
10. a common fraction
11. a decimal
12. a percentage
13. a ratio

Activity 11. Replace one word or phrase in each sentence with the pronoun "one".

1. If a person enjoys solving puzzles, the person might find satisfaction in mathematical challenges.
2. If a student encounters difficulties, the student can seek assistance from the teacher.
3. If you find a red apple, please give me the red apple.
4. If you're looking for a good book, I recommend this book.
5. In a group discussion, it's important to consider everyone's ideas and choose the best idea.
6. In times of uncertainty, it's natural for a person to seek guidance from trusted sources.
7. It's always good to have a backup plan; no one wants to be caught without a backup plan.
8. Learning a new language can be challenging, but with dedication, a person can become fluent in a new language.
9. When attending a party, it's customary to bring a gift for the host, even if it's a small gift.
10. When faced with a difficult decision, trust your instincts; they can guide you to the right decision.

Activity 12. Match the definitions of the word "figure" (a–h) with the sentences (1–8).

a. a geometric shape
b. a human shape
c. a particular amount expressed as a number, especially a statistic
d. a person, especially an important one
e. an illustration or diagram in a text
f. another name for a digit
g. to calculate or compute
h. to think or guess

1. How can you **figure** the volume of cylinder?
2. If we can **figure** roughly how much it will cost, we can decide what to do.
3. It would be difficult to carry out an eight-**figure** calculation without a computer.
4. Leonard Euler is a distinguished **figure** in mathematics.
5. Similar **figures** have many geometric properties in common.
6. The illustration of the model is shown in **Figure** 2.
7. The latest unemployment **figures** are discouraging.
8. There before him stood a tall **figure** in black.

Activity 13. Research the subject "Establishment of Zero as a Number". Report your findings with a multimedia presentation.

Checklist for a multimedia presentation:

1. Remember that the visuals (presentation) should only be used to support the presenter's message.
2. The key word to prepare an effective presentation is **MINIMUM**:
   - make the necessary **minimum** number of slides;
   - reduce text to a **minimum** (proper names, dates, figures, graphs etc.), try not to write complete sentences;
   - use bullet points to structure information (as in Point 1);
   - keep design and content simple.
3. Remember the rule of six:
   - a maximum of six lines per slide,
   - a maximum of six words per line.
4. Check whether the visual really shows what you are saying.
5. Make sure your audience can read the visual (font size and colours).
6. Find effective headlines.
7. The last slide REFERENCES shows the list of sources you used to prepare your report.
8. When reporting speak to the audience and don't read the slides.

Unit 2. Numeracy

Activity 14. Complete the sentences with the words in the box.

calculate / count / equals / measure / solve

1. If something __________ a particular number or amount, it is the same as that amount or the equivalent of that amount.
2. If you __________ a number or amount, you discover it from information that you already have, by using arithmetic, mathematics, or a special machine.
3. If you __________ a quantity that can be expressed in numbers, such as the length of something, you discover it using a particular instrument or device, for example a ruler.
4. If you __________ a problem or a question, you find a solution or an answer to it.
5. When you __________, you say all the numbers one after another up to a particular number.

Activity 15. Choose the best alternative.

1. An equation / inequality like 8 < 12 indicates that 8 is less than 12.
2. Equality / Inequality in education is a fundamental principle that promotes fairness.
3. I used a calculate / calculator to determine the average of the test scores.
4. I will solve / solution the equation 3x + 7 = 22 to find the value of x.
5. Some items, like grains of sand, are practically countable / uncountable due to their vast quantity.
6. The calculable / incalculable number of grains of sand on the beach makes it impossible to count precisely.
7. The calculate / calculation of the perimeter involves adding the lengths of all sides.
8. The counting / countless exercise revealed that there are ten pencils on the desk.
9. The discount on the sale items is easily calculable / incalculable using a percentage.
10. The equal / unequal distribution of resources can lead to social disparities.

Activity 16. Read the passage. In pairs, discuss the questions in the box. When discussing you can use the connective words given in Appendix I. Cohesive Devices.

Numeracy is defined as the ability to understand and use maths in daily life, at home, work, or school; it doesn't mean complex skills, like algebra, it means being confident enough to use basic maths in real-life situations. Numeracy is not always taught in the classroom: it means having the confidence and skills to use maths to solve problems in everyday life. Numeracy is as important as literacy — it's sometimes called “mathematical literacy” — and we need both to get on in life. Numeracy means understanding how maths is used in the real world and being able to apply it to make the best possible decisions. It’s as much about thinking and reasoning as about “doing sums”. It means being able to: interpret data, charts, and diagrams; process information; solve problems; check answers; understand and explain solutions; make decisions based on logical thinking and reasoning.

(from National Numeracy)

1. What is numeracy?
2. What does it mean to be numerate? innumerate?
3. Is numeracy a basic or special skill? Why?
4. What are the benefits of numeracy?
5. What are the challenges of innumeracy?

Activity 17. Match the synonymous items in the two columns.

1. an area	a. to use
2. an upfront response	b. to increase greatly
3. diverse applications	c. to establish
4. the impact of	d. to be front and centre
5. to be efficient	e. to be essential for
6. to be integral to	f. to be effective
7. to be pervasive	g. the effect of
8. to determine	h. different uses
9. to employ	i. a field
10. to rise exponentially	j. a direct answer

Activity 18. Read the article to enumerate the applications of mathematics described in the text. Extend the list with your ideas. For that, use Appendix I. Cohesive Devices.

"When will you ever need to use maths in real life?" It's the question asked perennially in maths classrooms up and down the country and indeed around the world. The answer is easy — perhaps we should be responding to the question with the counter, "When will you not?"

The pandemic has provided some very upfront answers. For the last few years, we have been hearing regularly about the potential for cases to rise exponentially. News programmes carried regular features on the reproduction number, R. Others reported, in vain, that we might nearly have reached elusive mathematically defined herd immunity thresholds.

We relied on mathematical models, not only to understand the current situation but to predict what might happen in the future, from the impact of mitigations to the effectiveness of vaccines. We used maths to determine the most efficient order to deliver jabs during the vaccine rollout and to plan the roadmap out of lockdown in early 2021. Maths was front and centre much of the time.

Even outside times of crisis we see maths in the newspaper headlines every day. We use it to establish whether our politicians are telling the truth about unemployment. Maths allows us to monitor exchange rates during currency crashes. It is invaluable to opinion pollsters determining the popularity of our political parties and to fact-checkers holding politicians to account.

Away from the front-page headlines, maths is the language of science. It appears everywhere from physics to engineering and chemistry — aiding us in understanding the origins of the universe and building bridges that won't collapse in the wind. Perhaps a little more surprisingly, maths is also increasingly integral to biology. Scientists in my own specialist area of mathematical biology are helping to develop treatments for diseases and to answer the question of how the leopard got its spots.

Beyond the academy, we are increasingly employing maths in sport to enhance the performance of our top athletes. We use it in the movies to create computer-generated images of scenes that couldn't exist in reality. More mundanely, we are frequently using maths in our everyday lives when we go shopping or when we follow a recipe, when we tell the time or when we budget for the future. Much of the time we do it without even realising it.

Certainly, much of the maths we learn early on in school we use directly in our everyday lives. Other topics that we might have learnt later, or perhaps we never got around to, are essential for the functioning of modern society even if we don't often see their use directly.

There are of course bits of maths (particularly pure maths) for which it is harder to imagine a direct use. But isn't this true of every subject? Should we hold geography, for example, to the same exacting standards of utility we expect of maths? I don't remember the last time I put my hard-won knowledge of oxbow lakes to use. Similarly, in chemistry, when was the last time you needed to write down the chemical reaction diagram depicting esterification? Probably not very recently.

This is not to denigrate these subjects, but to point out that this is not a maths-specific issue. Perhaps maths suffers more because it is harder to visualise the direct application of an algebraic equation than it is to picture the flow of water in a river, for example. We can all remember sitting by a river watching the water flow past, but fewer of us, I would suggest, can imagine laying down our picnic blanket on the complex plane of an Argand diagram.

By necessity, maths tends to deal in generalities and therefore abstractions from reality. But, at least in part, it is the generality — the abstractness — which makes mathematics so pervasive.

At university, I teach students that a single abstract equation can describe the spread of heat through your radiator, the diffusion of a drop of food colouring in a glass of water and the random dispersion of cells on a petri dish. With such a diverse range of applications, you can start to see how powerful it is to study a seemingly abstract and lifeless equation for the deep insights it can provide about ostensibly unrelated systems.

It was not for nothing that philosopher Eugene Wigner wrote of "the unreasonable effectiveness of mathematics" for describing the natural world. Many simple mathematical ideas come up over and over again in different areas. The "normal distribution" — or bell curve — for example, can be used to describe people's IQs as well as their heights and has hundreds of other applications too. The problem mathematics faces may be that it has too many applications.

(by Kit Yates, from The Independent, 2022)

Activity 19. Watch the video "Why Do People Get So Anxious About Math?" to choose the best answer to the questions. Does being bad at mathematics render one unintelligent? Why? Watch the video again and make a note of all the myths and stereotypes associated with mathematics. Disprove them. When expressing disagreement, use Appendix I. Cohesive Devices. Disagreeing.

https://disk.yandex.ru/i/iqnZm-k2Yh-Hrg

1. What happens to working memory when someone experiences math anxiety?

A. It increases and helps solve problems faster
B. It decreases because worry uses it up
C. It stays the same but becomes more efficient
D. It completely stops functioning during tests

2. Why does math anxiety seem to be more common than anxiety in other subjects?
A. Math is naturally harder than all other subjects
B. Only intelligent people can experience math anxiety
C. How parents and teachers present math to children influences their anxiety levels
D. Students spend more time studying math than other subjects

3. What do relaxation techniques and writing down worries have in common as strategies for math anxiety?
A. They both require a teacher's supervision
B. They both help free up working memory to focus on math
C. They both take several hours to be effective
D. They both work only for elementary school students

4. What does the "growth mindset" principle suggest about mathematical ability?
A. Only children can develop their math skills
B. Math skills are determined at birth and cannot change
C. The brain areas for math can develop and improve over time
D. Growth mindset only works for people who are already good at math

5. What advice does the text give to teachers and parents of young children?
A. Focus on speed and quick problem-solving
B. Separate boys and girls during math lessons
C. Be playful with math and allow time for children to work through answers
D. Tell children that math is challenging so they take it seriously

Activity 20. Label the points related to mathematical anxiety as either a cause, symptom, effect, or solution.

1) avoiding everyday situations involving maths at work or at home, like helping children with homework;
2) being in pressured situations, such as fearing being judged on how quickly you can produce an answer, or sitting an exam;
3) blocking any motivation to practise in order to learn and progress;
4) creating or amplifying a belief that maths ability is "fixed" and cannot be improved;
5) easing into it, working at your own pace, without the pressure to master a problem straight away;

6) feeling flustered, panicked or stressed, experiencing sweating and nausea, having increased heart rate;
7) having cultural bias, for example implications from opinions in the media and popular culture that because of background or gender someone is likely to have lower ability in maths;
8) having reduced performance in some situations and tests;
9) having specific negative past experiences, for example having felt humiliated for getting something wrong while in school;
10) leaving the individual caught in a cycle of anxiety;
11) making the time, just ten minutes here and there, to give sums a go, ideally somewhere relaxed, so it doesn't feel like a test environment;
12) overcoming maths myths;
13) preventing people from applying for courses, jobs, and promotions;
14) recognising the emotion and that it won't always be this way, i.e. that this is the way that you feel now, but not forever;
15) remembering that the ability to be good at maths isn't something we are born with; it can change over time, and we can all be good with numbers;
16) setting achievable goals, which feel reachable;
17) struggling to concentrate on a calculation;
18) talking it through, looking online, or asking a colleague or friend what they would do.

Activity 21. In groups, discuss the points.

1. Reflect on personal experiences with mathematical anxiety. Can you relate to situations where confidence in mathematical ability varies depending on the context? Share your own experiences if applicable.
2. Discuss the phenomenon of imposter syndrome in the context of mathematical anxiety. How does the fear of not measuring up to others impact one's capacity to think logically, especially in public spaces?
3. Explore the potential impact of mathematical anxiety on life decisions and career choices. How might a person's avoidance of mathematical situations influence their professional path, and what are the implications for educational and career opportunities?
4. Investigate the cyclic nature of mathematical anxiety and its potential causes. How do negative experiences with mathematics contribute to anxiety, and what factors may be under the control of educators to mitigate these issues?
5. Consider long-term solutions to address mathematical anxiety, such as focused professional development for teachers and changing the prevailing "can't do" culture. Discuss the feasibility and potential impact of these solutions on creating a more positive learning environment.

6. Explore the concept of mathematical resilience. How can fostering a "can do" attitude and a growth mindset contribute to overcoming mathematical anxiety? Share strategies that promote a positive idea of resilience in mathematics.
7. Examine the gender differences in the prevalence of mathematical anxiety, particularly its higher occurrence in girls. How do social and historical factors contribute to these differences, and what steps can be taken to encourage interest in maths among all students, regardless of gender?
8. Discuss the function of role models, particularly in the context of female mathematicians. How can role models contribute to breaking down stereotypes and encouraging a positive attitude toward mathematics, especially among girls?

Activity 22. Write a paragraph on the role that mathematics plays in day-to-day life. Follow the plan:

1. Start by clearly stating the main idea. (e.g., "*Mathematics plays a crucial and often unseen role in our daily lives*.").
2. Provide 2–3 concrete examples from everyday life. (e.g., *budgeting, cooking, shopping, time management, sports*).
3. Briefly explain how math is used. (e.g., "*We use ratios and proportions when following a recipe*.").
4. Use linking words to make your writing smooth (e.g., *For example, Furthermore, Another area is, In conclusion*).
5. End with a sentence that summarizes your point and shows why it's important.

Unit 3. Defining Mathematics

Activity 23. Complete the spaces with the correct form of the words from the box.

branch / calculate / count / mathematician / mathematics / measure / problem

(1) __________ is arguably the oldest of the sciences. It began with man's need to (2) __________ objects and to (3) __________ distances. A (4) __________ uses numbers and signs to (5) __________ fixed quantities or to compute variable quantities.

Mathematics is known as the most exact of all the sciences since the proper use of its methods can provide only one correct answer to a specific (6) __________. It is the language used by all the other sciences. It is the basis for precision in such (7) __________ as astronomy, chemistry, and physics.

Activity 24. Choose the best alternative.

1. The equality / equation 2x + 5 = 11 can be solved to find the value of x.
2. The immeasurable / measurable vastness of the universe captivates our imagination.
3. The impact of the discovery was immeasurably / measurably significant for scientific progress.
4. The measure / measurement of the rectangle's sides indicated it was a square.
5. The mystery remained solved / unsolved, leaving the investigators puzzled.
6. The number of stars in the sky seems countable / countless on a clear night.
7. The solve / solution to the problem is 15, as confirmed by my calculations.
8. The solved / unsolved puzzle revealed a beautiful image of a rainbow.
9. The weights of the two boxes are equal / equate.
10. Using a computer program, it is easy to calculate / count the square root of any given number.

Activity 25. In pairs, discuss the questions.

1. How would you define mathematics?
2. Why do you think some people state that it is hard to define mathematics?
3. Which descriptor(s) and why would you use to define mathematics: (a) an innate ability; (b) a human construct; (c) a language; (d) a natural phenomenon; (e) a philosophy; (f) a religion; (g) a science; (h) a tool; (i) an art; (j) the truth?

Activity 26. Read the article to collect all the definitions of mathematics given in the text. Classify them as either formal or informal.

Believing myself to be a non-mathematician, my dormant interest in maths has been awakened by the lively discussions going on around me every day, and the environment in which I am steeped has prompted me to ask the question "What is mathematics?" My search for the answer has unearthed some interesting ideas which I'd like to share.

In her blog on mathematics and ethics Lucy Rycroft-Smith says that mathematics is "the language of pattern, measurement and logical rules". Now the idea of mathematics being a language resonates with me, evoking the memory of a statement that mathematicians the world over can communicate and understand each other through this "metalanguage par excellence".

In a blog on the role of paradox in mathematics, Vinay Kathotia stated that "mathematics is the art of interpreting, quantifying, and working with error and uncertainty". This gave me pause — I had never thought of maths as an art before. I have marvelled at the art produced by mathematics — for instance, the boundless beauty of fractals or the complexity of my son's computer-designed creations — but to think of mathematics itself as an art, and especially as an art which deals with error and uncertainty was beyond my imagining. Hadn't I always been taught that maths was certain, more like a science based on logic and facts, and that errors were wrong, resulting in red crosses all over my exercise book?

Moving on to a computer search the inevitable Wikipedia provided the less-than-helpful (although no doubt accurate) "Mathematics has no generally accepted definition", before looking at a variety of suggested definitions ranging from Aristotle ("the science of quantity"), through abstract and philosophical definitions ("symbolic logic"; "carrying out mental constructions"; "the examination of the properties and interactions of idealized objects"), to humorous or even poetical definitions ("the art of giving the same name to different things" — there's that word art again).

A more thorough online search however led me to these wonderful words by Dr Liaqat Khan, Professor in Mathematics at Quaid-i-Azam University, Islamabad: "Mathematics is concerned with using imagination, intuition and reasoning to find new ideas and to solve puzzling problems”. Whilst his use of the qualifier “concerned with” means this is not a definition per se, the overarching vision he expresses is inspiring and excitingly inclusive. If we accept this statement, then we are all born mathematicians; we are inherently mathematical beings whose defining human characteristics (as opposed to those we share with other animals) are those which also make us mathematical, and I can no longer claim to be a non-mathematician.

What is more, no-one should be viewed by others as a non-mathematician, especially by those whose aim is to teach mathematics. If mathematics is a language, an art, a science, a philosophy, as well as innate, and if it embraces those very human traits of error and uncertainty, then learning mathematics should be delightful for all of the budding linguists, artists, scientists, philosophers and yes, even career mathematicians in each classroom. The challenge, of course, is finding ways of nurturing this delight — the imagination and intuition as well as the reasoning — within the constraints of a set curriculum.

(by Lynn Fortin, from Cambridge Mathematics, 2018)

Activity 27. Complete the chart with as many definitions of mathematics you can find using additional resources. Include your critical evaluation of each definition in the Notes column. Select or, if needed, formulate the optimum definition, and explain your rationale.

№	Author	Bio	Definition	Notes
1	Aristotle	Ancient Greek philosopher (4th century B.C.E.)	the science of quantity	*incomplete*
…				

Activity 28. Do you believe in the idea of a mathematical gene, that a mathematical ability is genetic? Dissect the weight that nature (genetics) and nurture (upbringing and education) carry in the process of developing mathematical knowledge and skills. Debate in groups.

Activity 29. Choose one quote and comment on it in writing.

1. “If I were again beginning my studies, I would follow the advice of Plato and start with mathematics.” (Galileo Galilei)
2. “Life is good for only two things, discovering mathematics and teaching mathematics.” (Siméon Denis Poisson)

Unit 4. Language of Mathematics

Activity 30. Complete the sentences with the phrases by combining the items in the two columns. Use the initial letters as clues. Some forms need changing.

1) achieve 2) carry out 3) conduct 4) do 5) find 6) find the answer to 7) make 8) perform 9) produce 10) reach 11) satisfy 12) solve 13) undertake 14) work out	a) calculation b) equation c) operation d) problem e) research f) solution

1. After careful consideration, we were able to r__________ a satisfactory s__________.
2. After hours of effort, I finally a__________ a s__________ to the complex equation.
3. Can you w__________ o__________ the e__________ for the area of the triangle?
4. He is skilled at helping students s__________ complex math p__________.
5. I always enjoy d__________ math p__________ in my spare time.
6. I need to d__________ a c__________ to determine the total cost.
7. It took me a while to s__________ the e__________, but I finally got the answer.
8. It took some time, but I finally f__________ the a__________ to the p__________.
9. I finally managed to w__________ o__________ a s__________ to the problem.
10. Let's m__________ a c__________ to figure out the average.
11. Let's work together to f__________ a s__________ to this challenging problem.
12. Mathematicians c__________ r__________ to push the boundaries of mathematical knowledge.
13. Scientists often d__________ r__________ to explore new mathematical theories.
14. She decided to u__________ r__________ to f__________ innovative s__________.
15. The scientist will c__________ o__________ o__________ on the data to analyze the results accurately.
16. The team collaborated to p__________ a s__________ that satisfies all requirements.
17. The values I found s__________ the e__________ perfectly.
18. To find the area of the rectangle, we need to p__________ the o__________ of multiplying its length by its width.

Activity 31. In pairs, discuss the questions.

1. What is meant by the language of mathematics?
2. Is mathematical language international? Why?
3. Is mathematical language universal? Why?
4. Do you find the language of mathematics to be easy or difficult? Why?

Activity 32. Label the concepts as algebraic or geometric.

1. circumference	5. imaginary number	9. perimeter	13. product
2. cone	6. multiplication	10. pi	14. quarter
3. cube	7. obelus	11. polygon	15. side
4. foot	8. parallelogram	12. problem	16. triangle

Activity 33. Read the article and match the headings (a–f) to the passages (1–6).

(a) International Rules
(b) Language as a Teaching Tool
(c) The Argument Against Math as a Language
(d) Vocabulary, Grammar, and Syntax in Mathematics
(e) What Is a Language?
(f) Why Mathematics Is a Language

(1) ____

Mathematics is called the language of science. Italian astronomer and physicist Galileo Galilei is attributed with the quote, "Mathematics is the language in which God has written the universe." Most likely this quote is a summary of his statement in Opere Il Saggiatore:

"[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word."

Yet, is mathematics truly a language, like English or Chinese? To answer the question, it helps to know what language is and how the vocabulary and grammar of mathematics are used to construct sentences.

In order to be considered a language, a system of communication must have vocabulary, grammar, syntax, and people who use and understand it.

Mathematics meets this definition of a language. Linguists who don't consider math a language cite its use as a written rather than spoken form of communication.

Math is a universal language. The symbols and organization to form equations are the same in every country of the world.

(2) ____

There are multiple definitions of "language." A language may be a system of words or codes used within a discipline. Language may refer to a system of communication using symbols or sounds. Linguist Noam Chomsky defined language as a set of sentences constructed using a finite set of elements. Some linguists believe language should be able to represent events and abstract concepts.

Whichever definition is used, a language contains the following components:

1. There must be a vocabulary of words or symbols.

2. Meaning must be attached to the words or symbols.

3. A language employs grammar, which is a set of rules that outline how vocabulary is used.

4. A syntax organizes symbols into linear structures or propositions.

5. A narrative or discourse consists of strings of syntactic propositions.

6. There must be (or have been) a group of people who use and understand the symbols.

Mathematics meets all of these requirements. The symbols, their meanings, syntax, and grammar are the same throughout the world. Mathematicians, scientists, and others use math to communicate concepts. Mathematics describes itself (a field called meta-mathematics), real-world phenomena, and abstract concepts.

(3) ____

The vocabulary of math draws from many different alphabets and includes symbols unique to math. A mathematical equation may be stated in words to form a sentence that has a noun and a verb, just like a sentence in a spoken language. For example:

$3 + 5 = 8$

could be stated as "Three added to five equals eight."

Breaking this down, nouns in math include:

1. Arabic numerals (0, 5, 123.7)

2. Fractions (1⁄4, 5⁄9, 2 1⁄3)

3. Variables (a, b, c, x, y, z)

4. Expressions (3x, x2, 4 + x)

5. Diagrams or visual elements (circle, angle, triangle, tensor, matrix)

6. Infinity (∞)

7. Pi (π)

8. Imaginary numbers (i, -i)

9. The speed of light (c)

Verbs include symbols including:

1. Equalities or inequalities (=, <, >)

2. Actions such as addition, subtraction, multiplication, and division (+, -, x or *, ÷ or /)

3. Other operations (sin, cos, tan, sec)

If you try to perform a sentence diagram on a mathematical sentence, you'll find infinitives, conjunctions, adjectives, etc. As in other languages, the role played by a symbol depends on its context.

(4) ____

Mathematics grammar and syntax, like vocabulary, are international. No matter what country you're from or what language you speak, the structure of the mathematical language is the same.

Formulas are read from left to right.

The Latin alphabet is used for parameters and variables. To some extent, the Greek alphabet is also used. Integers are usually drawn from i, j, k, l, m, n. Real numbers are represented by a, b, c, α, β, γ. Complex numbers are indicated by w and z. Unknowns are x, y, z. Names of functions are usually f, g, h.

The Greek alphabet is used to represent specific concepts. For example, λ is used to indicate wavelength and ρ means density.

Parentheses and brackets indicate the order in which the symbols interact.

The way functions, integrals, and derivatives are phrased is uniform.

(5) ____

Understanding how mathematical sentences work is helpful when teaching or learning math. Students often find numbers and symbols intimidating, so putting an equation into a familiar language makes the subject more approachable. Basically, it's like translating a foreign language into a known one.

While students typically dislike word problems, extracting the nouns, verbs, and modifiers from a spoken/written language and translating them into a mathematical equation is a valuable skill to have. Word problems improve comprehension and increase problem-solving skills.

Because mathematics is the same all over the world, math can act as a universal language. A phrase or formula has the same meaning, regardless of another language that accompanies it. In this way, math helps people learn and communicate, even if other communication barriers exist.

(6) ____

Not everyone agrees that mathematics is a language. Some definitions of "language" describe it as a spoken form of communication. Mathematics is a written form of communication. While it may be easy to read a simple addition statement aloud (e.g., 1 + 1 = 2), it's much harder to read other equations aloud (e.g., Maxwell's equations). Also, the spoken statements would be rendered in the speaker's native language, not a universal tongue.

However, sign language would also be disqualified based on this criterion. Most linguists accept sign language as a true language. There are a handful of dead languages that no one alive knows how to pronounce or even read anymore.

A strong case for mathematics as a language is that modern elementary-high school curricula uses techniques from language education for teaching mathematics. Educational psychologist Paul Riccomini and colleagues wrote that students learning mathematics require "a robust vocabulary knowledge base; flexibility; fluency and proficiency with numbers, symbols, words, and diagrams; and comprehension skills."

(by Anne Marie Helmenstine, from ThoughtCo, 2019)

Activity 34. Study Table 16. Relate and specify its levels to the stages of education.

Table 16. OECD Numeracy Framework

Below Level 1	At this level individuals must be able to carry out simple processes such as: • counting and sorting; • performing basic arithmetic operations with whole numbers and money; • recognising common spatial representations; • other familiar contexts where the mathematical content is clear with little or no text.
Level 1	At this level individuals must be able to carry out simple, one-step mathematical processes where the mathematical content is clear with little text, such as the following: • counting and sorting; • performing basic arithmetic operations; • understanding simple percentages; • locating and identifying simple, common graphical, or spatial representations.
Level 2	At this level individuals must be able to identify and act on mathematical information embedded in a range of common contexts where the mathematical content is fairly clear or visual. Tasks tend to require the application of two or more steps, such as the following: • processes involving calculation with whole numbers and common decimals; • percentages and fractions; • simple measurement and spatial representation; • estimation; • interpretation of relatively simple data and statistics in texts, tables, and graphs.
Level 3	At this level individuals must be able to understand mathematical information that may be less clear, embedded in contexts that are not always familiar and represented in more complex ways. Tasks require several steps and may involve the choice of problem-solving strategies, such as the following: • application of number sense and spatial sense; • recognising and working with mathematical relationships, patterns, and proportions expressed in verbal or numerical form; • interpretation and basic analysis of data and statistics in texts, tables, and graphs.
Level 4	At this level individuals must be able to understand a broad range of mathematical information that may be complex, abstract, or embedded in unfamiliar contexts.

	These tasks involve undertaking multiple steps and choosing relevant problem-solving strategies, such as the following: • analysis and more complex reasoning with quantities and data; • statistics and probability; • spatial relationships; • change, proportions and formulas; • understanding arguments and communicating explanations for answers.
Level 5	At this level individuals must be able to understand complex representations, abstract and formal mathematical and statistical ideas, possibly embedded in more complex forms. This includes the following: • integrating multiple types of mathematical information where considerable interpretation is required; • coming to well-reasoned conclusions; • working with mathematical arguments and models; • justifying, evaluating, and critically reflecting on answers.

Activity 35. In groups, construct a general model of mathematical competency, both knowledge-wise and skill-wise. Include mathematical language competence from the text in Activity 33. Share your ideas with other groups using a multimedia presentation.

Activity 36. Speak on the subject "Mathematics as a Universal Language and the Language of the Universe". As separate paragraphs, include the introduction with your thesis statement, the body with your arguments, and the conclusion.

Unit 5. Branches of Mathematics

Activity 37. Use one and the same word to complete all the sentences. Define the meaning of the word in each sentence.

1. A __________ of mathematics known as geometry focuses on the properties and relationships of shapes.
2. He chose to __________ out in his career, exploring opportunities in different fields.
3. The bank opened a new __________ in the city to expand its services.
4. The company decided to __________ into international markets to expand its business.
5. The detective carefully examined every __________ of the investigation to solve the complex case.
6. The family tree showed how each __________ represented a different generation.
7. The oak tree had a thick __________, providing shade on a sunny day.
8. The river split into a smaller __________, creating a picturesque scene in the forest.

Activity 38. Complete the chart with the names of scientific fields.

<table>
<tr><td colspan="5">anthropology / astronomy / biology (life science) / chemistry / computer science / Earth science / economics / engineering / history / information science / law / linguistics / logic / mathematics / medicine / pedagogy / physics / political science / psychology / sociology</td></tr>
<tr><th colspan="4">Fundamental Sciences</th><th rowspan="2">Applied Sciences</th></tr>
<tr><th colspan="2">Natural Sciences</th><th>Social Sciences</th><th>Formal Sciences</th></tr>
<tr><th>Physical Science</th><td rowspan="2"></td><td rowspan="2"></td><td rowspan="2"></td><td rowspan="2"></td></tr>
<tr><td></td></tr>
</table>

Activity 39. In pairs, discuss the questions.

1. Is mathematics a natural science, a social science, or a fundamental science? Why?
2. What subdisciplines constitute mathematics?
3. What areas of mathematics do you lean towards most? least? Why?
4. What areas of mathematics do you find most challenging? least challenging? Why?

Activity 40. Read the article to differentiate between pure mathematics and applied mathematics, conceptually and historically.

The study of abstract mathematical systems and structures, without necessarily having practical applications in mind, is called pure mathematics. It has various branches, including abstract algebra, geometry, number theory, calculus, topology, and the topics derived from them. The study and use of the mathematical techniques to solve practical problems is called applied mathematics. The field has various branches including statistics, probability, mechanics, mathematical physics.

The distinction between pure mathematics and applied mathematics might not be sharp. For example, Euclidean geometry could be analyzed as an abstract study of the relationships between lines, points, and geometric shapes based on the foundations of Euclid's postulates, or could, at the same time, be viewed as a study of results that could potentially (and, in fact, has proved to be) useful to architects, surveyors, engineers, and scientists. Or, the general study of vectors and vector spaces can be viewed as either an abstract study or a practical one if one later has in mind to use this theory to analyze force diagrams in mechanics.

Although much of the mathematics developed in the time of antiquity was clearly motivated by practical concerns, the development of mathematics for its own sake was nonetheless of interest to early scholars. For instance, Babylonian tablets from ca. 1600 B.C.E. list large Pythagorean triples that could have no practical use. Greek mathematicians of around 400 B.C.E. began to seek rigour, proof, and justification in their mathematical thinking, and ca. 300 B.C.E. Euclid produced his logically rigorous treatise "The Elements", summarizing all mathematical knowledge known at his time. The unique organization of ideas presented in his work became the key feature of the piece. That, in itself, was seen as an analysis of logical thinking, one that became the paradigm of all mathematical and scientific thinking for the two millennia that followed.

During the 19th century, mathematicians began to search for unifying ideas between distinct branches of algebra and geometry. The general study of structures and operations on them led to the development of abstract algebra, for instance. The development of paradoxes in set theory and in the foundations of calculus forced scholars to seek greater levels of rigour and abstraction. Even the nature of logical reasoning itself was examined as an attempt to understand and resolve fundamental paradoxes. The need for abstract analysis and synthesis was recognized as important, and dichotomy between applied and pure mathematics became more apparent.

(from Encyclopaedia Britannica)

Activity 41. Identify the branches of mathematics based on the descriptions.

algebra / analysis / analytic geometry / applied mathematics / arithmetic / calculus / differential calculus / Euclidean geometry / geometry / group theory / integral calculus / logic / non-Euclidean geometry / number theory / probability theory / pure mathematics / set theory / statistics / topology / trigonometry

1. The branch of algebra that deals with mathematical groups.
2. The branch of calculus concerned with the determination of integrals and their application to the solution of differential equations, the determination of areas and volumes, etc.
3. The branch of calculus concerned with the study, evaluation, and use of derivatives and differentials.
4. The branch of geometry describing the properties of a figure that are unaffected by continuous distortion, such as stretching or knotting.
5. The branch of geometry that uses algebraic notation and analysis to locate a geometric point in terms of a coordinate system.
6. The branch of mathematics concerned with numerical calculations, such as addition, subtraction, multiplication, and division.
7. The branch of mathematics concerned with the foundations of mathematics.
8. The branch of mathematics concerned with the properties and interrelationships of sets.
9. The branch of mathematics concerned with the properties of trigonometric functions and their application to the determination of the angles and sides of triangles.
10. The branch of mathematics concerned with the properties, relationships, and measurement of points, lines, curves, and surfaces.
11. The branch of mathematics consisting of calculus and its higher developments.
12. The branch of mathematics in which arithmetical operations and relationships are generalized by using alphabetic symbols to represent unknown numbers or members of specified sets of numbers.
13. The branch of mathematics, developed independently by Newton and Leibniz.
14. The branch of modern geometry in which certain axioms of Euclidean geometry are restated.

15. The calculation, description, manipulation, and interpretation of the mathematical attributes of sets or populations too numerous or extensive for exhaustive measurements.
16. The geometry based on the definitions and axioms set out in "The Elements".
17. The mathematical study of probability.
18. The study and use of the mathematical techniques to solve practical problems.
19. The study of abstract mathematical systems and structures, without necessarily having practical applications in mind.
20. The study of integers, their properties, and the relationship between integers.

Activity 42. Match the fields of mathematics with the key figures they are associated with.

1. algebra	a. Blaise Pascal
2. analytic geometry	b. Euclid
3. calculus	c. Évariste Galois
4. geometry	d. Georg Cantor
5. group theory	e. George Boole
6. mathematical logic	f. Gottfried Wilhelm Leibniz and Sir Isaac Newton
7. mathematical statistics	g. Henri Poincaré
8. number theory	h. Hipparchus
9. probability theory	i. Sir Ronald A. Fisher
10. set theory	j. Muhammad ibn Musa al-Khwarizmi
11. topology	k. Pierre de Fermat
12. trigonometry	l. René Descartes

Activity 43. In groups, choose one subject in the box to brainstorm. Exchange your ideas with other groups. Follow the instructions below.

1. Mathematics as a Basic Life Skill
2. Mathematics as a Compulsory School Subject
3. Mathematics as an Advanced University Course
4. Mathematics as a Fundamental Formal Science
5. Mathematics as an Active Research Area

The basics of brainstorming

Brainstorming is a group problem-solving method that involves the spontaneous contribution of creative ideas and solutions. This technique requires intensive, freewheeling discussion in which every member of the group is encouraged to think aloud and suggest as many ideas as possible based on their diverse knowledge.

It is a method of generating ideas and sharing knowledge to solve a particular problem, in which participants are encouraged to think without interruption. Brainstorming is a group activity where each participant shares their ideas as soon as they come to mind.

Brainstorming rules

1. Focus on quantity. You've likely heard the phrase “quality over quantity,” but when it comes to brainstorming, the exact opposite is true.
2. Withhold criticism. Negativity has no place in a brainstorming session.
3. Welcome unusual ideas.
4. Combine and improve ideas.

"Mathematics is the queen of sciences and number theory is the queen of mathematics."

(Carl Friedrich Gauss)

Module 2. Number Theory and Algebra

Unit 6. Arithmetic

Activity 44. Watch the short film entitled "Alternative Math" to choose the best answer to the questions. What could be meant by the phrase "alternative math"? Watch the video again and make a note of all the mathematical words used in the film. Explain the meaning behind the title of the film.

https://disk.yandex.ru/i/PUh9ZN6OdBOKyA

Watch the video

1. What was Danny's mistake on the math test?
A. He didn't complete the test
B. He wrote the numbers next to each other instead of adding them
C. He refused to answer the question
D. He added the numbers incorrectly and got five

2. Why did Danny's parents come to the school?
A. To thank Mrs. Wells for helping their son
B. To discuss Danny's excellent performance
C. To complain about Mrs. Wells saying Danny's answer was wrong
D. To ask for extra homework for Danny

3. What did the principal suggest Mrs. Wells should do about the situation?
A. Give Danny a better grade
B. Apologize to the parents
C. Call the police
D. Transfer Danny to another class

4. What can we infer about the principal's attitude towards education?
A. He believes teachers should maintain academic standards
B. He thinks parents' opinions are more important than correct answers
C. He supports Mrs. Wells completely
D. He wants to improve the math curriculum

5. What happened to Mrs. Wells at the end of the story?
A. She received a promotion
B. She was given more students to teach
C. She lost her job at the school
D. She was asked to teach a different subject

Activity 45. Notice the two different ways of pronouncing the word "arithmetic" depending on its part of speech. Practise reading aloud the sentences.

aRITHmetic (noun) arithMETic (adjective) = arithmetical

1. **Arithmetic** is the branch of mathematics that deals with the study of numbers and the basic operations of addition, subtraction, multiplication, and division.
2. Children often start learning **arithmetic** in elementary school to build a strong foundation in mathematical concepts.
3. Mastery of **arithmetic** is essential for success in more advanced branches of mathematics.
4. Mental **arithmetic** involves performing calculations in one's mind without the use of external aids.
5. The **arithmetic** calculations were straightforward, requiring only basic mathematical operations.
6. The **arithmetic** exercise book contained a variety of problems to test the students' computational skills.
7. The **arithmetic** lesson covered fundamental mathematical principles, preparing students for more advanced coursework.
8. The **arithmetic** problems became more complex as the students progressed through the curriculum.
9. The **arithmetic** skills of the students were evident as they effortlessly solved the mathematical problems.
10. The teacher focused on teaching **arithmetic** to enhance the students' numerical abilities.

Activity 46. Role-play the conversation between Alice, the White Queen, and the Red Queen from Lewis Carroll's "Through the Looking-Glass". Find the names of arithmetical operations.

"Manners are not taught in lessons," said Alice. "Lessons teach you to do sums, and things of that sort."

"And you do addition?" the White Queen asked. "What's one and one and one and one and one and one and one and one and one and one?"

"I don't know," said Alice. "I lost count."

"She can't do addition," the Red Queen interrupted. "Can you do subtraction? Take nine from eight."

"Nine from eight I can't, you know," Alice replied very readily: "but —"

"She can't do subtraction," said the White Queen. "Can you do division? Divide a loaf by a knife — what's the answer to that?"

"I suppose —" Alice was beginning, but the Red Queen answered for her. "Bread-and-butter, of course. Try another subtraction sum. Take a bone from a dog: what remains?"

Alice considered. "The bone wouldn't remain, of course, if I took it — and the dog wouldn't remain; it would come to bite me — and I'm sure I shouldn't remain!"

"Then you think nothing would remain?" said the Red Queen.

"I think that's the answer."

"Wrong, as usual," said the Red Queen: "the dog's temper would remain."

"But I don't see how —"

"Why, look here!" the Red Queen cried. "The dog would lose its temper, wouldn't it?"

"Perhaps it would," Alice replied cautiously.

"Then if the dog went away, its temper would remain!" the Queen exclaimed triumphantly.

Alice said, as gravely as she could, "They might go different ways." But she couldn't help thinking to herself, "What dreadful nonsense we are talking!"

"She can't do sums a bit!" the Queens said together, with great emphasis.

“Can *you* do sums?” Alice said, turning suddenly on the White Queen, for she didn’t like being found fault with so much.

The Queen gasped and shut her eyes. “I can do addition, if you give me time — but I can’t do subtraction, under any circumstances!”

(from “Through the Looking-Glass, and What Alice Found There,” by Lewis Carroll, 1871)

Activity 47. Read the dialogue between Alice and the Mock Turtle from Lewis Carroll’s “Alice in Wonderland”. Explain the puns in bold.

“I only took the regular course.”

“What was that?” inquired Alice.

“**Reeling** and **Writhing**, of course, to begin with,” the Mock Turtle replied; “and then the different branches of Arithmetic — **Ambition**, **Distraction**, **Uglification**, and **Derision**.”

“I never heard of ‘**Uglification**,’” Alice ventured to say. “What is it?”

(from “Alice’s Adventures in Wonderland,” by Lewis Carroll, 1865)

Figure 2. The Gryphon and the Mock Turtle (by Sir John Tenniel)

Activity 48. In pairs, discuss the questions.

1. What are the three Rs?
2. What role do the three Rs play in education?
3. What are the four basic operations of elementary arithmetic?
4. Is mental arithmetic (mental calculation) an essential skill? Why?
5. Why is it integral to have the multiplication table (times table) memorized?

Activity 49. Read the article to discriminate between arithmetic and number theory.

The branch of mathematics concerned with computations using numbers is called arithmetic. This can involve a number of specific topics — the study of operations on numbers, such as addition, multiplication, subtraction, division, and square roots, needed to solve numerical problems; the methods needed to change numbers from one form to another (such as the conversion of fractions to decimals and vice versa); or the abstract study of the number systems, number theory, and general operations on sets as defined by group theory and modular arithmetic, for instance.

The word arithmetic comes from the Greek word “arithmetiké”, constructed from “arithmós” meaning “number” and “techné” meaning “science.” In the time of ancient Greece, the term “arithmetic” referred only to the theoretical work about numbers, with the word “logistic” used to describe the practical everyday computations used in business. Today the term “arithmetic” is used in both contexts.

The study of the arithmetic properties of numbers is called number theory. The fact that many simple statements about numbers can be extraordinarily difficult to prove, if at all possible, makes this topic an alluring and stimulating subject for mathematicians. (Goldbach’s conjecture, for instance, remains unsolved.)

Elementary number theory is the study of those topics in number theory that utilize only the basic techniques of arithmetic and high-school mathematics in their solutions. For example, the classification of the Pythagorean triples would be considered a problem in elementary number theory, as would the solution of many Diophantine equations. (The use of the word “elementary” here by no means implies that the level of mathematical sophistication used is elementary.) Analytic number theory incorporates the notion of limit in the study of numbers, and algebraic number theory extends the study of number theory to a general study of algebraic numbers and new number systems that include solutions to otherwise unsolvable algebraic equations.

(from Encyclopaedia Britannica)

Activity 50. Study the tables.

Table 17

<table>
<tr><th colspan="5">Addition</th></tr>
<tr><td colspan="2">1 + 2 = 3</td><td colspan="3">one plus two equals three
one and two is/make/give three</td></tr>
<tr><td>addend
summand
augend</td><td>plus sign</td><td>addend
summand</td><td>equal(s) sign
equality sign</td><td>sum</td></tr>
<tr><td>1</td><td>+</td><td>2</td><td>=</td><td>3</td></tr>
</table>

Table 18

<table>
<tr><th colspan="5">Subtraction</th></tr>
<tr><td colspan="2">3 – 2 = 1</td><td colspan="3">three minus two equals one
two from three is one</td></tr>
<tr><td>minuend</td><td>minus sign</td><td>subtrahend</td><td>equal(s) sign
equality sign</td><td>difference</td></tr>
<tr><td>3</td><td>–</td><td>2</td><td>=</td><td>1</td></tr>
</table>

Table 19

<table>
<tr><th colspan="5">Multiplication</th></tr>
<tr><td colspan="2">2 × 3 = 6
2 · 3 = 6</td><td colspan="3">two multiplied by three equals six
two times three is six</td></tr>
<tr><td>factor
multiplier</td><td>multiplication sign</td><td>factor
multiplicand</td><td>equal(s) sign
equality sign</td><td>product</td></tr>
<tr><td>2</td><td>x (times sign)
· (raised dot)</td><td>3</td><td>=</td><td>6</td></tr>
</table>

Table 20

<table>
<tr><th colspan="6">Division</th></tr>
<tr><td colspan="2">6 ÷ 3 = 2
7 ÷ 3 = 2 (1)</td><td colspan="2">6/3 = 2
7/3 = 2 (1)</td><td colspan="2">six divided by three equals two
seven divided by three equals two, remainder one</td></tr>
<tr><td>dividend</td><td>division sign</td><td>divisor</td><td>equal(s) sign
equality sign</td><td>quotient</td><td>remainder</td></tr>
<tr><td>6</td><td>÷ (obelus) / (slash)</td><td>3</td><td>=</td><td>2</td><td></td></tr>
<tr><td>7</td><td>÷ (obelus) / (slash)</td><td>3</td><td>=</td><td>2</td><td>1</td></tr>
</table>

Table 21

Exponentiation (Power)			
$3^2 = 9$ three squared equals nine three (raised) to the power of two equals nine three (raised) to the second (power) equals nine the second power of three is nine		**$2^3 = 8$** two cubed equals eight two (raised) to the power of three equals eight two (raised) to the third (power) equals eight the third power of two is eight	
base	**exponent** **power** **index**	**equal(s) sign** **equality sign**	**power**
3	^{2}	=	9
2	^{3}	=	8

Table 22

Extraction (Root)				
$\sqrt{4}$ **= 2** the square root of four is two $\sqrt[3]{8}$ **= 2** the cube root of eight is two $\sqrt[4]{16}$ **= 2** the fourth root of sixteen is two				
degree **index**	**radical sign** **root sign**	**radicand**	**equal(s) sign** **equality sign**	**root**
^{2}	√	4	=	2
^{3}	√	8	=	2
^{4}	√	16	=	2

Table 23

Equality			
=	is equal to	a = b	A is equal to B
Inequation (Inequality)			
≠	is not equal to	a ≠ b	A is not equal to B
Inequality			
>	is greater than	a > b	A is greater than B
<	is less than	a < b	A is less than B
≥	is greater than or equal to	a ≥ b	A is greater than or equal to B
≤	is less than or equal to	a ≤ b	A is less than or equal to B

Activity 51. In pairs, calculate the expressions and read them out loud. Give the names of the elements.

Addition	Subtraction
1) 9 + 3 = 2) 27 + 436 = 3) 4 + 36 + 19 = 4) 236 + 782 = 5) 5,345 + 655 = 6) 2 + 1 + 38 + 3 + 6 = 7) 4,447 + 7,478 + 676 = 8) 32,812 + 65,034 + 54,323 = 9) –6 + 3 = 10) –72 + (–73) = 11) 8 + (–6) + (–9) + 5 + 1 = 12) (–31 + 12) + (3 + (–16)) = 13) –24 + (–3) + 24 + (–5) + 5 =	14) 148 – 87 = 15) 343 – 269 = 16) 10,435 – 10,218 = 17) 5,231 – 5,177 = 18) 7,800 – 5,725 = 19) –7 – 6 = 20) –7 – (–6) = 21) 82 – (–109) = 22) 0 – 15 = 23) –60 – 50 – 40 =
Multiplication	**Division**
24) 49 × 9 = 25) 5 × 7 × 6 = 26) 72 × 10,000 = 27) 110 × 440 = 28) 157 · 59 = 29) 3,723 · 46 = 30) 5,624 · 281 = 31) 502 · 459 = 32) 8 × 0 = 33) 7 × 1 = 34) –10 × 7 = 35) –4(–73) = 36) –4(2) (–6) = 37) –9(–3) (–1) (–2) = 38) –20,000(1,300) =	39) 72 ÷ 4 = 40) 595 ÷ 35 = 41) 1,443 ÷ 39 = 42) 20,876 ÷ 68 = 43) 1,269 ÷ 54 = 44) 405 ÷ 21 = 45) 0 ÷ 10 = 46) 5,347 ÷ 127 = 47) 1,482,000 ÷ 3,900 = 48) –5 ÷ 1 = 49) 0 ÷ (–6) = 50) –18 ÷ (–18) =

Activity 52. In pairs, calculate the expressions and read them out loud.

Power	Root
1) 1^9	16) $\sqrt{12}$
2) 2^5	17) $\sqrt{121}$
3) 2^8	18) $\sqrt{144}$
4) 3^4	19) $\sqrt{16}$
5) 3^7	20) $\sqrt[3]{1}$
6) 4^5	21) $\sqrt[3]{125}$
7) 5^2	22) $\sqrt[3]{216}$
8) 5^3	23) $\sqrt[3]{27}$
9) 6^1	24) $\sqrt[4]{256}$
10) 6^3	25) $\sqrt[4]{81}$
11) 7^2	26) $\sqrt{49}$
12) 7^3	27) $\sqrt{64}$
13) 8^2	28) $\sqrt{81}$
14) 9^1	29) $\sqrt{9}$
15) 9^2	

Activity 53. In pairs, put an inequality sign and read the statements out loud.

1) 9 __ 7
2) 301 __ 310
3) 0 __ –7
4) –20 __ –19
5) –8 __ –9
6) –213 __ 123
7) –5 __ 0
8) 5.68 __ 5.75
9) 106.8199 __ 106.82
10) –78.23 __ –78.303
11) –555.098 __ –555.0991

Activity 54. Create ten expressions using addition, subtraction, multiplication, division, exponentiation, and extraction as well as five inequalities. Read them aloud for your partner to write down symbolically. Check if the expressions are correct. Swap roles.

Activity 55. Complete the divisibility rules.

1. All numbers are divisible by __.
2. A number is divisible by __ only if its final digit is 0, 2, 4, 6 or 8.
3. A number is divisible by __ if its final two digits represent a two-digit number that can be divided by 2 twice.
4. A number is divisible by __ only if its final digit is 0 or 5.
5. A number is divisible by __ only if it is an even number whose digits sum to a multiple of 3.
6. A number is divisible by __ if its final three digits represent a three-digit number that can be divided by 2 three times.
7. A number is divisible by __ only if its final digit is a zero.
8. A number is divisible by __ only if it is divisible by both 3 and 4.

Activity 56. Decipher the mnemonics used to memorize the order of precedence.

1. BEDMAS
2. PEMDAS

Activity 57. Complete the table with the properties of operations.

associative property / commutative property / distributive property	
Property	**Operations**
(1) ___________	a + (b + c) = (a + b) + c a × (b × c) = (a × b) × c
(2) ___________	a + b = b + a a × b = b × a
(3) ___________	a × (b + c) = a × b + a × c

Activity 58. Design an informative and illustrated classroom poster on the basic operations of elementary arithmetic, their order of precedence, and properties.

Unit 7. Numeral Systems and Base Systems

Activity 59. Complete the table with the Hindu-Arabic equivalents of the Roman numerals.

Roman	I	V	X	L	C	D	M
Hindu Arabic							

Activity 60. Match the corresponding numerals in the two columns.

(1) 2,000
(2) 4
(3) 6
(4) 13
(5) 17
(6) 27
(7) 29
(8) 44
(9) 60
(10) 90
(11) 173
(12) 360
(13) 444
(14) 600
(15) 800
(16) 999
(17) 1170
(18) 1250

(a) CCCLX
(b) CDXLIV
(c) CLXXIII
(d) CMXCIX
(e) DC
(f) DCCC
(g) IV
(h) LX
(i) MCCL
(j) MCLXX
(k) MM
(l) VI
(m) XC
(n) XIII
(o) XLIV
(p) XVII
(q) XXIX
(r) XXVII

Activity 61. Watch the video "A Brief History of Numerical Systems" to choose the best answer to the questions. What systems of numeration are you aware of? Expand on the revolutionary nature of the place-value system and positional notation.

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1. What was the main problem with early number systems like Greek, Hebrew, and Egyptian numerals?
A. They could only represent numbers up to one hundred
B. They required repeating symbols many times and creating new symbols for larger numbers
C. They were too difficult for merchants to understand
D. They could not represent the number zero at all

2. How does positional notation improve upon earlier number systems?
A. It uses the same symbols in different positions to represent different values
B. It requires fewer people to learn how to count properly
C. It eliminates the need for mathematical calculations
D. It works only with the number ten and its multiples

3. Why was the invention of zero considered a key breakthrough?
A. It allowed people to count backwards for the first time
B. It made mathematics easier to teach in schools
C. It prevented confusion between numbers like 63 and 603
D. It replaced all other symbols in the number system

4. Why do most number systems use base 10?
A. Because it is the most mathematically efficient system
B. Because ancient mathematicians proved it was superior
C. Because humans have ten fingers
D. Because it was required by early governments

5. What advantage does a base 12 system have over base 10?
A. It uses fewer symbols to write large numbers
B. It can be divided evenly by more numbers
C. It is easier for children to learn
D. It works better with modern computers

Activity 62. Read the article to outline the system of Roman numerals and that of Hindu-Arabic numerals.

More than 5,000 years ago an Egyptian ruler recorded, perhaps with a bit of exaggeration, the capture of 120,000 prisoners, 400,000 oxen, and 1,422,000 goats. This event was inscribed on a ceremonial mace which is now in a museum in Oxford, England.

The ancient Egyptians developed the art of counting to a high degree, but their system of numeration was very crude. For example, the number 1,000 was symbolized by a picture of a lotus flower and the number 2,000 was symbolized by a picture of two lotus flowers growing out of a bush. Although these symbols, called hieroglyphics, permitted the Egyptians to write large numbers, the numeration system was clumsy and awkward to work with. The number 999, for instance, required 27 individual marks.

In our system of numeration, we use ten symbols called digits — 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 — and combinations of these symbols. Our system of numeration is called decimal, or base-ten, system. There is little doubt that our ten fingers influenced the development of a numeration system based on ten digits.

The ancient Hindus are credited with discovering the decimal system of numeration we use today. This system was translated into Arabic prior to its introduction into Europe by travelling merchants around the 13th century. Hence it is also known as the Hindu-Arabic system. Adoption of the Hindu-Arabic system met resistance due to the widespread use of the Roman numeral system during this period.

Based on a simple tally system similar to the one used by the ancient Egyptians, merchants of the Roman empire of about 500 B.C.E. used letter symbols for powers of 10 and for the intermediate values of 5, and simply grouped symbols together to represent all other quantities.

The expression CLXXIII, for instance, represented the number 100 + 50 + 10 + 10 + 1 + 1 + 1 = 173. Although the order of the symbols was not important, it became the convention to list symbols from largest to smallest, left to right.

Initially the symbols D and M were not part of the Roman system. The number 1,000 was written (I), and further applications of round brackets allowed for the expression of even greater quantities. For instance, ((I)) represented 10,000, and (((I))) represented 100,000. Stonemasons introduced the symbols D and M to simplify their work.

The Romans also introduced other ornamentations to increase the value of a numerical symbol. For instance, vertical bars were used to represent a 100-fold increase, and a bar placed above the symbol represented a 1,000-fold increase.

There was no symbol for zero in the Roman system. To avoid the four-fold repetition of symbols (as in the expression CCCCXXXXIIII for 444), a subtractive principle was introduced in the 13th century:

The placement of a small value immediately to the left of a higher value indicates that that small value is to be subtracted from the higher value.

Thus, 4 could be written as IV, 90 as XC, and 444 as CDXLIV. The subtractive principle was subject to two rules:

1. *The symbols V, L, and D cannot be used as the numbers to be subtracted.*
2. *Only one symbol I, X, or C can be placed before a higher number symbol.*

Thus, for example, it was not permissible to write IIX for eight. Although not a proper place-value system, with the subtractive principle in use, the order of the symbols used was now important.

Performing operations of basic arithmetic with Roman numerals is very awkward. For example, it is not immediate what the solution to the following addition problem should be:

XLIV + XVII + XXIX

That European merchants were comfortable working with the Roman numeral system for well over a millennium suggests that scholars did not use the numeral system to perform calculations, only to record the results. (Arithmetic was performed on a counting board such as an abacus.)

The system of Roman numerals remained popular in Western Europe until the 17^{th} century. Although the system was eventually replaced by the Hindu-Arabic numeral system we use today, it still remains a tradition to use Roman numerals for numbering introductory pages in books, for instance.

Using a base-10 place-value system, numbers in the Hindu-Arabic system are expressed via combinations of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, organized so as to represent groupings of powers of 10. For instance, the number 574 represents the five groups of 100, seven groups of 10, and four single units.

This numerical system originated from India around 600 C.E., almost in the exact same form as we use it today. The system was transmitted to the Arabs two centuries later as they worked to translate the Sanskrit works on astronomy into Arabic. The Arab mathematician Muhammad ibn Musa al-Khwarizmi (ca. 800) wrote an influential treatise describing the Hindu numeral system and used it in his famous book "Calculation by Restoration and Reduction", from whose title the modern word "algebra" is derived. As Western scholars began translating the Arabic texts into Latin, word of the efficient numeration system spread across Western Europe. The Italian scholar Fibonacci (ca. 1170–1250) avidly promoted their use. By the end of the 17^{th} century, the Hindu-Arabic numeral system completely replaced the cumbersome system of Roman numerals that were the standard in Europe for over 1,500 years.

Other numeration systems were developed in early cultures and societies. Two of the most common were the base-five system, related to the number of fingers on one hand and the base-twenty system, related to the number of fingers and toes.

In some languages the word for “five” is the same as the word for “hand”, and the word for “ten” is the same as the word for “two hands”. In the English language the word “digit” is a synonym for the word “finger” — that is, ten digits, ten fingers.

Still another early system of numeration was a base-sixty system developed by the Mesopotamians and used for centuries. These ancient people divided the years into 360 days (6 × 60); today we still divide the hour into 60 minutes and the minute to 60 seconds. Numeration systems of current interest include a binary, or base-two, system used in electronic computers and a base-twelve, or duodecimal system.

(from Elementary Algebra)

Activity 63. Reorder the sentences to make a text on numeral systems.

a. A place-value system, such as the Arabic numeral system, has clear advantages in economy of symbolism and in efficiency of computation.
b. For example, in 333, the 3 on the right means three, but the 3 in the middle means three tens and the 3 on the left means three hundreds.
c. In modern Roman numerals (where I, V, X, L, C, D, and M are 1, 5, 10, 50, 100, 500, and 1,000, respectively), on the other hand, CCC means 300 — each C stands for one hundred, and the relative position of the C's is of no importance.
d. In this system the position a symbol occupies helps determine the value of the symbol.
e. Only the 10 symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 and the decimal point are needed to write numbers of any size.
f. The numeration system (or system for writing number symbols) widely used throughout the world today is a place-value system based on the number 10 and usually called the Arabic, or Hindu-Arabic, numeration system.
g. There are some rules regarding the order of symbols in the Roman numeral system, however (for example, IX means 9, while XI means 11), though generally position is not as important as in place-value systems.

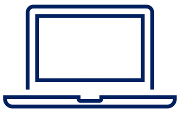
Activity 64. Develop a system of criteria for comparing numeral systems. Using it, contrast the Hindu-Arabic numerical system with the Roman system of numeration. Argue for the widespread use of the former in today's world.

Activity 65. Draw a mind map showing the application of several base systems of your choice.

Table 24. Base Systems

System	Base Value	System	Base Value	System	Base Value
		undenary	11		
binary	2	duodecimal	12	vigesimal	20
ternary	3				
quaternary	4				
quinary	5				
senary	6	hexadecimal	16	sexagesimal	60
septenary	7				
octal	8				
nonary	9				
decimal	10				

Unit 8. Number Sets



Activity 66. Complete the sentences with the plural form of the words in brackets.

1. ___________ (hyperbola) take different shapes in analytic geometry.
2. ___________ (matrix) simplify complex calculations in linear algebra.
3. ___________ (polyhedron) include figures like cubes and prisms.
4. ___________ (rhombus) are a type of parallelograms.
5. ___________ (torus) exhibit unique topological properties.
6. Calculate the areas of these ___________ (trapezium).
7. Earth's rotation involves multiple ___________ (axis).
8. Ensure proper use of ___________ (parenthesis) in mathematical expressions.
9. Mathematicians rely on many ___________ (lemma) to prove theorems.
10. Natural ___________ (phenomenon) are studied in physics.
11. Postgraduate students defend their ___________ (thesis).
12. Roll the ___________ (die) to simulate different outcomes.
13. The ___________ (analysis) of the data reveal interesting patterns.
14. The ___________ (locus) of points form intricate patterns.
15. The book explores captivating ___________ (series) of mathematical ideas.
16. The converging lines meet at various ___________ (vertex).
17. The event marked significant changes over several ___________ (millennium).
18. Mathematical ___________ (proof) confirm the validity of theorems.
19. The mountain range has several towering ___________ (apex).
20. The scientists conducted lots of ___________ (research) on the topic.
21. The scientists derived different ___________ (formula) for their experiments.
22. The scientists tested several ___________ (hypothesis) to explain the phenomenon.
23. The success ___________ (criterion) for the project are diverse.
24. Various circles have distinct ___________ (radius).

Activity 67. Watch the video “Making Sense of Irrational Numbers” to choose the best answer to the questions. Describe how Hippasus’ discovery revolutionized mathematics.

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1. What was Hippasus' "crime" according to the Greek myth?
A. He murdered his guests at a religious ceremony
B. He discovered irrational numbers through mathematical proof
C. He challenged Pythagoras to a public debate
D. He stole sacred mathematical texts from the gods

2. What did the Pythagorean mathematicians believe about numbers?
A. Numbers were useful tools for counting objects
B. Numbers were the building blocks of the universe and could all be expressed as ratios
C. Numbers were created by the gods and shouldn't be studied
D. Numbers were less important than geometry and shapes

3. How did Hippasus prove that the square root of 2 could not be expressed as a ratio?
A. He used a proof by contradiction to show that assuming it was rational led to an impossible situation
B. He calculated the decimal expansion and showed it never ended
C. He measured the diagonal of a square and found it was impossible
D. He asked other mathematicians and they all agreed with him

4. What is the key point about irrational numbers like the square root of 2 and pi?
A. They are mistakes in mathematics that should be avoided
B. They can eventually be expressed as ratios if we try hard enough
C. Decimals and ratios are just ways to express numbers, but these numbers have exact values
D. They are only theoretical and cannot be represented in any way

5. Why is forming right triangles on a number line mentioned?
A. To show that irrational numbers can be precisely plotted even though they can't be expressed as ratios
B. To prove that the square root of 2 is actually a rational number
C. To demonstrate that geometry is more important than algebra
D. To explain how the Pythagoreans originally discovered these numbers

Activity 68. In pairs, discuss the questions.

1. What do the letters N, Z, Q, R, C mean to a mathematician?
2. Define N, Z, Q, R, C in terms of subsets and supersets.
3. Represent the relations between N, Z, Q, R, C symbolically.

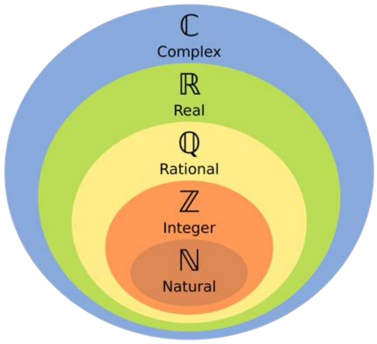

Figure 3. Number Systems

Activity 69. Read the article to complete the table based on the text.

The development of different types of numbers can be seen as motivated by the need for solving different types of equations. For example, the counting numbers (that is, the natural numbers N) suffice for solving any equation of the type x + 2 = 5, for instance, but not an equation of the type x + 5 = 2. (There is no solution to this equation within the set of counting numbers.) This motivates the introduction of negative numbers and the construction of the integers Z (from the German word “Zahlen” for “numbers”).

Working solely in the realm of the whole numbers, a number is said to be even if it is divisible by 2, and odd if it leaves a remainder of 1 when divided by 2. For example, 18 is divisible by 2 and so is even, and 23 leaves a remainder of 1 and so is odd. As the study of evenness and oddness shows, the number 0 is even. Two integers that are either both even or both odd are said to have the same parity. For instance, 17 and 53 have the same parity (both are odd), and 9 and 14 have opposite parity. Sometimes the term “parity” is used in a more general setting as to mean “being in one of two possible states” (either positive or negative).

A whole number possessing just two positive factors is called a prime number, or simply a prime. For example, 7 has only two positive factors, namely 1 and 7, and so is prime. The number 24 has eight positive factors and so is not prime, and the number 1 has only one factor and is not prime. The term composite is used to describe numbers greater than 1 that are not prime, that is positive whole numbers with more than two positive factors. (Medieval mathematician Fibonacci (1170–1250) called prime numbers “incomposite.”) It is vital that the number 1 be considered neither prime nor composite for the fundamental theorem of arithmetic to hold true.

But the set of integers is not always sufficient for solving equations of the type 5x = 3, for instance. Desiring solutions to equations of this type leads to the construction of fractions and the set of all rational numbers Q (for “quotient”). A rational number is any number that can be written in the form a/b, where “a” and “b” represent integers and $b \neq 0$. The set of rational numbers is the set of all terminating and all repeating decimals.

Unfortunately, again, not all equations can be solved within the rational system. For example, the equation $x^2 - 2 = 0$ has no rational solution. Extending the set of rational numbers to include solutions to equations of this type introduces irrational numbers and the construction of the real number system R.

An irrational number is a nonterminating, nonrepeating decimal. An irrational number cannot be expressed as a fraction with an integer numerator and a nonzero integer denominator. Two subsets of irrationals are algebraic and transcendental numbers. A number is called algebraic if it is the root of a polynomial with integer coefficients. For example, (1/2) $(5 + \sqrt{13})$ is algebraic since it is a solution to the equation $x^2 - 5x + 3 = 0$. Numbers that are not algebraic are called transcendental.

It is extraordinarily difficult to define precisely what is meant by a real number. Many standard texts in mathematics define a real number to be any rational number or any irrational number.

A real number “x” is said to be positive if it is greater than zero, that is, if $x > 0$. A real number less than zero is called negative. An unspecified real number that is positive or possibly zero is called nonnegative. One that is negative or possibly zero is called nonpositive. Zero is the only real number that is neither positive nor negative.

Yet the system of real numbers also does not suffice for solving all equations. With the introduction of a single additional number, denoted “i”, to represent an “imaginary” solution to the equation $x^2 + 1 = 0$, the complex numbers C are born. The number “i” is usually regarded as the square root of negative one: $i = \sqrt{-1}$. (One must be careful as there are, in fact, two square roots of this quantity, namely “i” and “–i”.) Surprisingly, as shown by the fundamental theorem of algebra, the introduction of this single number is all that is needed to solve any polynomial equation $a_n x^n + ... + a_1 x + a_0 = 0$. Thus, the complex numbers represent a system of numbers that is algebraically closed in the sense that the construction of no new type of number is needed to solve arithmetic equations.

On a conceptual level, the notion of “number” is intimately connected with the act of counting. Simple counting systems of ancient times used tally marks to record numbers, and over the millennia this basic numeration scheme evolved to the sophisticated place-value system we use today. (The ancient Egyptians of around 3000 B.C.E. were perhaps the first to move from the use of tally marks alone.) It was a great intellectual achievement for mankind when the notion of “number” was removed from the specific objects being counted, recognizing, for instance, that two cows, two houses, and two days all share a common property of “two-ness.” (Even today we sometimes use different words to count different types of “two.” For instance, the words “twins”, “couple”, and “pair” cannot be used interchangeably to represent two people.) This simple recognition of an abstract commonality between sets of objects was exploited by German mathematician Georg Cantor (1845–1918) who, in the late 1800s, developed a general notion of cardinality. With it, Cantor extended the notion of “number” to include counts of sets of infinite size. He established, for instance, that there are an infinite number of different types of infinity and managed to develop a meaningful system of arithmetic for his transfinite numbers.

The Irish mathematician Sir William Rowan Hamilton (1805–65) followed a different route and worked to extend the notion of “number” to represent operations on n-dimensional space. An Argand diagram shows that the complex numbers have a natural representation as points on a plane. Hamilton sought to give meaning to an arithmetic for points in three- and higher-dimensional space. Although he did not succeed in accomplishing this goal for three-dimensional space, his invention of the quaternions shows this feat can be done in four-dimensional space. (The octonions provide an arithmetic for eight-dimensional space.)

(from Elementary Algebra)

№	**Set**	**Type**	**Definition**	**Examples**
1	N	a natural number		
2		a whole number		
3		an even number		
4		an odd number		
5		a prime number		
6		a composite number		
7	Z	an integer		
8	Q	a rational number		
9		an irrational number		
10		an algebraic number		
11		a transcendental number		
12	R	a real number		
13		a positive number		
14		a negative number		
15	C	a complex number		

Activity 70. Complete the parity properties (even or odd).

1. The sum of two even numbers is ___________.
2. The sum of any number of even numbers is ___________.
3. The difference of two even numbers is ___________.
4. The sum of an even number and an odd number is ___________.
5. The difference of an even number and an odd number is ___________.
6. The sum of two odd numbers is ___________.
7. The sum of an even number of odd numbers is ___________.
8. The sum of an odd number of odd numbers is ___________.
9. The difference of two odd numbers is ___________.
10. The product of two even numbers is ___________.
11. The product of an even number and an odd number is ___________.
12. The product of two odd numbers is ___________.

Activity 71. Watch the video "A Brief History of Banned Numbers" to choose the best answer to the questions. Why do you think numbers can be banned? In your view, what numbers may have been banned? Compare your ideas to the ones in the video.

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1. What did the Pythagoreans believe about mathematics?
A. It was only useful for counting objects
B. It held the deepest secrets of the universe
C. It was less important than philosophy
D. It should be kept secret from everyone

2. Why were Hindu-Arabic numerals banned in Florence?
A. They were too difficult for merchants to learn
B. Religious leaders thought they were against God
C. Authorities worried they could be easily forged or altered
D. They made it impossible to record business transactions

3. What can be inferred about Hippasus?
A. He was rewarded for his mathematical discoveries
B. His discovery challenged the Pythagoreans' beliefs about the universe
C. He invented irrational numbers to prove his teachers wrong

D. He was the most famous mathematician in ancient Greece

4. Why might some numbers be illegal today?
A. They are too large for computers to process
B. They represent protected information like copyrights or state secrets
C. They are connected to ancient mathematical theories
D. Merchants use them to cheat their customers

5. What is the main idea of the video?
A. Mathematics has always been the most important subject in education
B. Throughout history, certain numbers have been considered dangerous or banned for various reasons
C. The Pythagoreans were wrong about their mathematical theories
D. Modern governments should stop making numbers illegal

Activity 72. Write a paragraph justifying the existence of multiple number systems.

Unit 9. Algebra

Activity 73. Complete the table with the types of algebraic expressions.

equality / equation / identity / inequality / inequation	
Expression	**Example**
(1) ___________	a = b
(2) ___________	$a \neq b$
(3) ___________	$x > y$
(4) ___________	3x + 4 = 8
(5) ___________	x + 5 + x = 2x + 5

Activity 74. Choose the best alternative.

1. Constants / Variables are letters (or symbols) that stand for numbers.
2. To perform the multiplication 3(x + 4), we use the associative / distributive property.
3. Terms such as $7x^2$ and $5x^2$, which have the same variables raised to exactly the same power, are called like / similar terms.
4. The coefficient / constant of the term 9y is 9.
5. To estimate / evaluate $y^2 + 9y – 3$ for y = –5, we substitute –5 for y and apply the order of operations rule.
6. An equality / equation is a statement indicating that two expressions are equal.
7. To solve an equation means to find all magnitudes / values of the variable that make the equation true.

Activity 75. Match the algebraic structures to the axioms they satisfy.

1. group (G) [4]
2. Abelian group [5]
3. ring (R) [7]
4. commutative ring [8]
5. integral domain [9]
6. field [10]

a. associative law of addition
b. associative law of multiplication
c. associativity
d. closure
e. commutative law for addition
f. commutative law for multiplication
g. commutativity
h. distributive laws
i. existence of a 1
j. existence of a zero
k. existence of additive inverses

l. existence of an identity
m. existence of inverses
n. existence of multiplicative inverses
o. no divisors of zero

Activity 76. Read the article. Expand on what constitutes the scope of modern algebra as a distinct branch of mathematics.

The branch of mathematics concerned with the general properties of numbers, and generalizations arising from those properties, is called algebra. Often symbols are used to represent generic numbers, thereby distinguishing the topic from the study of arithmetic. For instance, the equation 2 × (5 + 7) = 2 × 5 + 2 × 7 is a (true) arithmetical statement about a specific set of numbers, whereas the equation x × (y + z) = x × y + x × z is a general statement describing a property satisfied by any three numbers. It is a statement in algebra.

Much of elementary algebra consists of methods of manipulating equations to either put them in a more convenient form, or to determine (that is, solve for) permissible values of the variables that appear. For instance, rewriting $x^2 + 6x + 9 = 25$ as $(x + 3)^2 = 25$ allows an easy solution for "x": either x + 3 = 5, yielding x = 2, or x + 3 = –5, yielding x = –8.

The word "algebra" comes from the Arabic term "al-jabr" used by the great Muhammad ibn Musa al-Khwarizmi (ca. 780–850) in his writings on the topic.

In modern times the subject of algebra has been widened to include abstract algebra, group theory, and the study of alternative number systems such as modular arithmetic. Boolean algebra looks at the algebra of logical inferences, matrix algebra the arithmetic of matrix operations, and vector algebra the mechanics of vector operations and vector spaces.

An algebraic structure is any set equipped with one or more operations (usually binary operations) satisfying a list of specified rules. For example, any group, ring, field, or vector space is an algebraic structure.

Research in pure mathematics is motivated by one fundamental question: what makes mathematics work the way it does? For example, to a mathematician, the question, "What is 263 × 178 (or equivalently, 178 × 263)?" is of little interest. A far more important question would be, "Why should the answers to 263 × 178 and 178 × 263 be the same?"

The topic of abstract algebra attempts to identify the key features that make algebra and arithmetic work the way they do. For example, mathematicians have shown that the operation of addition satisfies five basic principles, and that all other results about the nature of addition follow from these.

1. *Closure: The sum of two numbers is again a number.*

2. *Associativity: For all numbers “a”, “b”, and “c”, we have: (a + b) + c = a + (b + c).*
3. *Zero element: There is a number, denoted “0,” so that: a + 0 = a = 0 + a for all numbers “a”.*
4. *Inverse: For each number “a” there is another number, denoted “–a,” so that: a + (–a) = 0 = (–a) + a.*
5. *Commutativity: For all numbers “a” and “b” we have: a + b = b + a.*

Having identified these five properties, mathematicians search for other mathematical systems that may satisfy the same five relations. Any fact that is known about addition will consequently hold true in the new system as well. This is a powerful approach to matters. It avoids having to re-prove theorems and facts about a new system if one can recognize it as a familiar one in disguise. For example, multiplication essentially satisfies the same five axioms as above, and so for any fact about addition, there is a corresponding fact about multiplication. The set of symmetries of a geometric figure also satisfy these five axioms, and so too all known results about addition immediately transfer to interesting statements about geometry. Any system that satisfies these basic five axioms is called an “Abelian group,” or just a group if the fifth axiom fails. Group theory is the study of all the results that follow from these basic five axioms without reference to a particular mathematical system.

The study of rings and fields considers mathematical systems that permit two fundamental operations (typically called addition and multiplication). Allowing for the additional operation of scalar multiplication leads to a study of vector spaces.

The theory of algebraic structures is highly developed. The study of vector spaces as well as matrices, for example, is so extensive that the topic is regarded as a field of mathematics in its own right and is called linear algebra. As matrices are used to analyze and solve systems of simultaneous linear equations and to describe linear transformations between vector spaces, this topic of study unites geometric thinking with numerical analysis. As the set of all invertible matrices of a given size form a group, called the general linear group, techniques of abstract algebra can also be incorporated into this work.

(from Elementary Algebra)

Activity 77. Study the tables.

Table 25

Axis Variable	**British**	**American**
x	/eks/	
y	/waɪ/	
z	/zed/	/zi/

Table 26

Algebraic Expression			
2x + 1 two X plus one			
Term			**Term**
Coefficient **Parameter**	**Variable** **Indeterminate** **Unknown**	**Operator**	**Constant** **Known**
2	x	+	1

Table 27

1.	**$x(y + z) = xy + xz$**
	X times the sum of Y and Z equals XY plus XZ
2.	**$ax^2 + bx^2 + c = 0$**
	AX squared plus BX squared plus C equals zero
3.	**$x^2 + 2px^2 + p^2 = (x + p)^2$**
	X squared plus two PX squared plus P squared equals the sum of X and P squared
4.	**$a(b + c) = ab + ac$**
	A times, bracket/parenthesis, B plus C, close bracket/parenthesis, equals AB plus AC
5.	**$(2a – a)/a = 1$**
	a. bracket/parenthesis, two A minus A, close bracket/parenthesis, divided by A equals one b. two A minus A ALL/QUANTITY divided by A equals one

Activity 78. While commenting on the steps, simplify and solve the equations and inequalities.

Equations	Inequalities
1) $3x – 8 – 4x – 7x = –2 – 8$ 2) $–6t – 7t – 5t – 1 = 12 – 3$ 3) $4(d – 5) + 20 = 5 – 2d$ 4) $1 – t = 5(t – 2) + 10$ 5) $30x – 12 = 1{,}338$ 6) $40y – 19 = 1{,}381$	19) $5x – 1 \leq 29$ 20) $5 + 4x > 25$ 21) $5(2 – x) \leq 30$ 22) $2x + 3x < 200 – 5x$ 23) $5(x + 2) < 6(9 – x)$ 24) $x^2 + 4x \leq x(x – 5) – 18$ 25) $x(x + 2) < x(2 – x) + 2x^2$

7) $-7 = \frac{3}{7}r + 14$	26) $2x + 3 \leq 17$
8) $21 = \frac{2}{5}f - 19$	27) $5 - 4x > 25$
9) $10 - 2y = 8$	
10) $7 - 7x = -21$	
11) $9 + 5(r + 3) = 6 + 3(r - 2)$	
12) $2 + 3(n - 6) = 4(n + 2) - 21$	
13) $-\frac{2}{3}z + 4 = 8$	
14) $-\frac{7}{5}x + 9 = -5$	
15) $-2(9 - 3s) - (5s + 2) = -25$	
16) $4(x - 5) - 3(12 - x) = 7$	
17) $9a - 2.4 = 7a + 4.6$	
18) $4c - 1.6 = 7c + 3.2$	

Activity 79. What is Blaise Pascal famous for? Watch the video "The Mathematical Secrets of Pascal's Triangle" to choose the best answer to the questions. Describe the practical applications of Pascal's triangle.

https://disk.yandex.ru/i/HIgpG34vTGDfkw

Figure 4. Blaise Pascal

1. What does the text suggest about Pascal's Triangle's origin?
A. Pascal was the first mathematician to discover it
B. It was independently discovered by mathematicians in different cultures
C. It was originally created in France and then spread to other countries
D. Pascal stole the idea from Indian mathematicians

2. How is each new row in Pascal's Triangle created?
A. By multiplying the numbers in the previous row by two
B. By adding consecutive pairs of numbers from the row above
C. By listing all the odd numbers in sequence
D. By squaring each number from the previous row

3. What happens when you add up all the numbers in any row of Pascal's Triangle?
A. You get a triangular number

B. You get a multiple of eleven
C. You get a power of two
D. You get a prime number

4. What is the main purpose of using Pascal's Triangle in probability problems?
A. To calculate how many different combinations are possible
B. To determine which outcome is most likely to occur
C. To predict future events based on past patterns
D. To create fractal patterns for statistical analysis

5. What is implied about Pascal's Triangle?
A. It has been completely understood and has no more secrets to reveal
B. It is only useful for solving basic mathematical problems
C. Mathematicians continue to discover new applications and patterns in it
D. It is too complex for practical use in modern mathematics

Unit 10. Development of Algebra

Activity 80. Complete the table with the types of equations.

a cubic equation / a linear equation / a quadratic equation / a quartic equation / a quintic equation	
Equation	**General Form ($a \neq 0$)**
(1) ___________	$ax + b = c$
(2) ___________	$ax^2 + b + c = 0$
(3) ___________	$ax^3 + bx^2 + cx + d = 0$
(4) ___________	$ax^4 + bx^3 + cx^2 + dx + e = 0$
(5) ___________	$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$

Activity 81. Do the quiz on the advancement of algebra. In pairs, compare your answers.

1. **When did the pursuit of finding solutions to equations begin?**
   a. the 3rd century B.C.E.
   b. 1650 B.C.E.
   c. 825 C.E.
   d. the 19th century
2. **What method did the ancient mathematicians, including the Babylonians, use to solve equations?**
   a. algebraic manipulation
   b. false position
   c. completing the square
   d. abstract reasoning
3. **What mathematical challenge did the Greeks face when dealing with cubic equations?**
   a. difficulty in solving linear equations
   b. trouble in geometric constructions for cubic products

c. lack of symbolic representation
d. inability to work with negative numbers

4. **When did symbols start being used in algebraic problems?**
   a. the 3rd century
   b. 825 C.E.
   c. the 16th century
   d. the 19th century
5. **Who is credited with making a significant step towards the development of modern algebra in the 9th century?**
   a. Fibonacci
   b. Diophantus
   c. Muhammad ibn Musa al-Khwarizmi
   d. Euclid
6. **What did al-Khwarizmi's work lead to, influencing the study of algebra in Europe?**
   a. the spread of symbolic logic
   b. the cossic art
   c. quadratic equation solutions
   d. the Rhind papyrus
7. **Which Renaissance scholar published solutions to cubic and quartic equations in "The Great Art"?**
   a. René Descartes
   b. Girolamo Cardano
   c. Scipione del Ferro
   d. Niccolò Tartaglia
8. **What did Carl Friedrich Gauss prove in 1797, shaping the understanding of algebra?**
   a. the commutative property
   b. the fundamental theorem of algebra
   c. the existence of irrational numbers
   d. the general formula for quintic equations
9. **What important characteristic of algebra changed with the introduction of abstract algebraic systems?**
   a. commutative multiplication
   b. geometric proofs
   c. symbolic notation
   d. negative numbers
10. **Who made significant contributions to the understanding of noncommutative algebras in the late 19th to early 20th century?**
    a. René Descartes
    b. Carl Friedrich Gauss
    c. Amalie Noether
    d. Paolo Ruffini

Activity 82. Read the article. Review your answers to the quiz in Activity 81.

Finding solutions to equations is a pursuit that dates back to the ancient Egyptians and Babylonians and can be traced through the early Greeks' mathematics. The Rhind papyrus, dating from around 1650 B.C.E., for instance, contains a problem reading:

A quantity; its fourth is added to it. It becomes fifteen. What is the quantity?

Readers are advised to solve problems like these by a method of "false position," where one guesses (posits) a solution, likely to be wrong, and adjusts the guess according to the result obtained. In this example, to make the division straightforward, one might guess that the quantity is four. Taking 4 and adding to it its fourth gives, however, only 4 + 1 = 5, one-third of the desired answer of 15. Multiplying the guess by a factor of three gives the solution to the problem, namely 4 × 3, which is 12.

Although the method of false position works only for linear equations of the form ax = b, it can nonetheless be an effective tool. In fact, several of the problems presented in the Rhind papyrus are quite complicated and are solved relatively swiftly via this technique.

Clay tablets dating back to 1700 B.C.E. indicate that Babylonian mathematicians were capable of solving certain quadratic equations by the method of completing the square. They did not, however, have a general method of solution and worked only with a set of specific examples fully worked out. Any other problem that arose was matched with a previously solved example, and its solution was found by adjusting the numbers appropriately.

Much of the knowledge built up by the old civilizations of Egypt and Babylonia was passed on to the Greeks. They took matters in a different direction and began examining all problems geometrically by interpreting numbers as lengths of line segments and the products of two numbers as areas of rectangular regions. Followers of Pythagoras from the period 540 to 250 B.C.E., for instance, gave geometric proofs of the distributive property and the difference of two squares formula, for example, in much the same geometric way we use today to explain the method of expanding brackets. The Greeks had considerable trouble solving cubic equations, however, since their practice of treating problems geometrically led to complicated three-dimensional constructions for coping with the product of three quantities.

At this point, no symbols were used in algebraic problems, and all questions and solutions were written out in words (and illustrated in diagrams). However, in the 3rd century, Diophantus of Alexandria introduced the idea of abbreviating the statement of an equation by replacing frequently used quantities and operations with symbols as a kind of shorthand. This new focus on symbols had the subtle effect of turning Greek thinking away from geometry. Unfortunately, the idea of actually using the symbols to solve equations was ignored until the 16th century.

The Babylonian and Greek schools of thought also influenced the development of mathematics in ancient India. The scholar Brahmagupta (ca. 598–665) gave solutions to quadratic equations and outlined general methods for solving systems of equations containing several variables. (He also had a clear understanding of negative numbers and was comfortable working with zero as a valid numerical quantity.) The scholar Bhaskara (ca. 1114–85) used letters to represent unknown quantities and, in working with quadratic equations, suggested that all positive numbers have two square roots and that negative numbers have no (meaningful) roots.

A significant step toward the development of modern algebra occurred in Baghdad, Iraq, in the year 825 when the Arab mathematician Muhammad ibn Musa al-Khwarizmi (ca. 780–850) published his famous piece “Calculation by Restoration and Reduction”. This work represents the first clear and complete exposition on the art of solving linear equations by a new practice of performing the same operation on both sides of an equation. For example, the expression $x - 3 = 7$ can be “restored” to $x = 10$ by adding three to both sides of the expression, and the equation $5x = 10$ can be “reduced” to $x = 2$ by dividing both sides of the equation by five. Al-Khwarizmi also showed how to solve quadratic equations via similar techniques. His descriptions, however, used no symbols, and like the ancient Greeks, al-Khwarizmi wrote everything out in words. Nonetheless, al-Khwarizmi’s treatise was enormously influential, and his new approach to solving equations paved the way for modern algebraic thinking. In fact, it is from the word “al-jabr” in the title of his book that our word “algebra” is derived.

Al-Khwarizmi’s work was translated into Latin by the Italian mathematician Fibonacci (ca. 1175–1250), and his efficient methods for solving equations quickly spread across Europe during the 13th century. The art of algebra became known in Europe as “the cossic art” (from the Italian word “cosa” for “thing”).

Renaissance scholars Scipione del Ferro (1465–1526) and Niccolò Tartaglia (ca. 1500–57) both knew how to solve cubic equations. In 1545 Girolamo Cardano (1501–76) published “The Great Art”, which included solutions to the cubic and quartic equations discovered by his assistant Ludovico Ferrari (1522–65).

By the end of the 17th century, mathematicians were comfortable performing the same sort of symbolic manipulations we practice today and were willing to accept negative numbers and irrational quantities as solutions to equations. The French mathematician François Viète (1540–1603) introduced an efficient system for denoting powers of variables and was the first to use letters as coefficients before variables, as in $ax^2 + bx + c$, for instance. (Viète also introduced the signs + and –, although he never used a sign for equality.) René Descartes (1596–1650) introduced the convention of denoting unknown quantities by the last letters of the alphabet, “x”, “y”, and “z”, and known quantities by the first, “a”, “b”, “c”. (This convention is now completely ingrained; when we see, for example, an equation of the form $ax + b = 0$, we assume, without question, that it is for “x” we must solve.)

The German mathematician Carl Friedrich Gauss (1777–1855) proved the fundamental theorem of algebra in 1797, which states that every polynomial equation of degree "n" has at least one and at most "n" (possibly complex) roots. His work, however, does not provide actual methods for finding these roots.

For the centuries that followed, mathematicians attempted to find a general arithmetic method for solving all quintic (fifth-degree) equations. Leonhard Euler (1707–83) suspected that the task might be impossible. Between the years 1803 and 1813, the Italian mathematician Paolo Ruffini (1765–1822) published a number of algebraic results that strongly suggested the same, and just a few years later the Norwegian mathematician Niels Henrik Abel (1802–29) proved that, indeed, there is no general formula that solves all quintic equations in a finite number of arithmetic operations. Of course, some degree-five equations can be solved algebraically. (Equation of the form $x^5 - a = 0$, for instance, have solutions $\sqrt[5]{x} = \sqrt{a}$.) In 1831 the French mathematician Évariste Galois (1811–32) completely classified those equations that can be so solved, developing work that gave rise to a whole new branch of mathematics today called group theory.

In the 19th century mathematicians began using variables to represent quantities other than real numbers. For example, English mathematician George Boole (1815–64) invented an algebra of symbolic logic in which variables represented sets, and the Irish scholar Sir William Rowan Hamilton (1805–65) invented algebraic systems in which variables represented vectors or quaternions.

With these new systems, important characteristics of algebra changed. Hamilton, for instance, discovered that multiplication was no longer commutative in his systems: a product a × b might not necessarily give the same result as b × a. This motivated mathematicians to develop abstract axioms to explain the workings of different algebraic systems. Thus, the topic of abstract algebra was born. One outstanding contributor in this field was German mathematician Amalie Noether (1883–1935), who made important discoveries about the nature of noncommutative algebras.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 83. Identify the individuals based on the descriptions from the text in Activity 82.

Amalie Noether / Bhaskara / Brahmagupta / Carl Friedrich Gauss / Diophantus of Alexandria / Évariste Galois / Fibonacci / François Viète / George Boole / Girolamo Cardano / Leonhard Euler / Ludovico Ferrari / Muhammad ibn Musa al-Khwarizmi / Niccolò Tartaglia / Niels Henrik Abel / Paolo Ruffini / René Descartes / Scipione del Ferro / Sir William Rowan Hamilton

1. The Arab mathematician who published "Calculation by Restoration and Reduction," pioneering modern algebraic thinking.
2. The assistant to Girolamo Cardano, discovered solutions to cubic and quartic equations.
3. An English mathematician who invented an algebra of symbolic logic.
4. A French mathematician who classified equations solvable by algebraic methods, contributing to group theory.
5. A French mathematician who introduced an efficient system for denoting powers of variables.
6. A German mathematician who made important discoveries about the nature of noncommutative algebras.
7. A German mathematician who proved the fundamental theorem of algebra.
8. Introduced the convention of denoting unknown quantities by the last letters of the alphabet.
9. Introduced the idea of abbreviating the statement of an equation with symbols.
10. An Irish scholar who invented algebraic systems with variables representing vectors or quaternions.
11. An Italian mathematician who published results suggesting the impossibility of a general formula for quintic equations.
12. An Italian mathematician who translated al-Khwarizmi's work into Latin, spreading algebraic methods across Europe.
13. A Mathematician who suspected the impossibility of finding a general arithmetic method for solving all quintic equations.
14. A Norwegian mathematician who proved the impossibility of a general formula for quintic equations.
15. Published "The Great Art," including solutions to cubic and quartic equations.
16. A Renaissance scholar who knew how to solve cubic equations.
17. A scholar who gave solutions to quadratic equations and outlined general methods for solving systems of equations.
18. A scholar who used letters to represent unknown quantities and made contributions to quadratic equations.

Activity 84. Rearrange the events in chronological order according to the text of Activity 82. Provide dates where possible.

a. Amalie Noether makes significant discoveries about noncommutative algebras.
b. Babylonian mathematicians solve quadratic equations using the method of completing the square.
c. Carl Friedrich Gauss proves the fundamental theorem of algebra.
d. Diophantus introduces the idea of using symbols to abbreviate equations.
e. Equations are solved using the method of "false position" on the Rhind papyrus.
f. Fibonacci translates al-Khwarizmi's work into Latin, spreading algebraic methods in Europe.
g. François Viète introduces efficient systems for denoting powers of variables, and René Descartes establishes conventions for denoting known and unknown quantities.
h. Greeks interpret numbers geometrically, using lengths of line segments and areas of rectangular regions.
i. Mathematicians, including Paolo Ruffini and Niels Henrik Abel, explore general methods for solving quintic equations.
j. Mathematicians, such as George Boole and Sir William Rowan Hamilton, introduce algebraic systems representing sets, vectors, and quaternions.
k. Muhammad ibn Musa al-Khwarizmi publishes "Calculation by Restoration and Reduction," introducing algebraic methods in solving linear equations.
l. Scholars like Scipione del Ferro, Niccolò Tartaglia, and Girolamo Cardano contribute to solving cubic and quartic equations.

Activity 85. Determine whether the statements are true or false by quoting from the text in Activity 82.

1. Al-Khwarizmi did not heavily rely on symbols in his descriptions, similar to the ancient Greeks.
2. Babylonian mathematicians employed the technique of completing the square to find solutions for specific quadratic equations.
3. Babylonian mathematicians lacked a universal approach to solve various quadratic equations.

4. Fibonacci invented algebraic systems involving variables representing vectors or quaternions.
5. George Boole introduced the convention of using the first letters of the alphabet for denoting unknown quantities.
6. In 1797, Carl Friedrich Gauss proved the fundamental theorem of algebra.
7. In the 3rd century, Diophantus of Alexandria brought in the practice of using symbols to condense the expression of an equation.
8. Symbols were not commonly utilized by Greek mathematicians in their algebraic problem-solving.
9. The Italian mathematician Fibonacci translated the writings of al-Khwarizmi into Greek, disseminating algebraic techniques throughout Europe.
10. The work of Muhammad ibn Musa al-Khwarizmi in 825 played a crucial role in shaping modern algebraic thought.

Activity 86. In groups, discuss the points. Refer to the text in Activity 82.

1. Discuss the historical methods employed by ancient Egyptians and Babylonians, as seen in the Rhind papyrus, for finding solutions to equations. How effective was the method of "false position" in solving complex problems?
2. Explore the mathematical contributions of Babylonian mathematicians in solving quadratic equations using the method of completing the square. How did their approach differ from the methods used by other ancient civilizations?
3. Examine the geometric focus of Greek mathematicians, particularly followers of Pythagoras, in solving problems. How did their emphasis on geometry present challenges in dealing with cubic equations?
4. Analyze the transition from geometric problem-solving to symbolic notation introduced by Diophantus of Alexandria in the 3rd century. How did the use of symbols impact Greek thinking and the solving of equations?
5. Investigate the pivotal role played by Muhammad ibn Musa al-Khwarizmi in the development of modern algebra. How did his work in Baghdad in 825 lay the foundation for solving linear and quadratic equations?
6. Trace the spread of algebraic methods in Europe during the 13th century, facilitated by the translation of al-Khwarizmi's work by Fibonacci. How did this dissemination contribute to the recognition of algebra as "the cossic art"?
7. Examine the advancements made by Renaissance scholars like Scipione del Ferro, Niccolò Tartaglia, and Girolamo Cardano in solving cubic and quartic equations. How did their contributions shape the understanding of algebra in the 16th century?
8. Discuss the evolution of symbolic manipulations and the acceptance of negative numbers and irrational quantities in the 17th century. How did mathematicians like François Viète and René Descartes contribute to these developments?

9. Explore Carl Friedrich Gauss's proof of the fundamental theorem of algebra in 1797. How did this theorem impact the understanding of polynomial equations, and what were its implications?
10. Investigate the attempts to find a general arithmetic method for solving quintic equations in the 19^{th} century. How did mathematicians like Paolo Ruffini, Niels Henrik Abel, and Évariste Galois contribute to the classification of solvable equations and the emergence of group theory?

Activity 87. Choose one point in Activity 86 and elaborate on it in writing. Refer to the text in Activity 82.

"There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world." (Nikolai Lobachevsky)

Module 3. Geometry

Unit 11. Analytic Geometry

Activity 88. Add the missing forms in the table and complete the sentences with the words.

Noun	Noun (Person)	Adjective	Adverb	Verb
(singular) analysis (plural)		1) 2)		

1. A skilled mathematician acts as an __________, utilizing mathematical tools and methods to dissect complex problems and provide solutions.
2. An __________ approach in mathematics involves breaking down complex problems into simpler components for a more detailed examination.
3. Collaborative efforts among __________ are common in mathematical research, where multiple perspectives contribute to a more comprehensive understanding of a problem.
4. In __________ geometry, mathematical relationships between geometric shapes are explored using coordinate systems, providing a powerful tool to study the properties of curves, lines, and shapes through algebraic equations.
5. Mathematicians conduct rigorous __________ of mathematical structures to gain deeper insights into their properties and behaviour.
6. Mathematicians often __________ data sets, functions, and mathematical structures to extract meaningful information and draw relevant conclusions.
7. The __________ skills of mathematicians are crucial for interpreting data, formulating theorems, and developing precise solutions to mathematical challenges.
8. The ability to __________ mathematical concepts from different angles is a hallmark of a proficient mathematician, leading to a more comprehensive understanding of the subject.

Activity 89. In pairs, discuss the questions.

1. What was René Descartes?
2. What is the origin of the word “Cartesian”?
3. What is René Descartes’ greatest contribution to mathematics?
4. What is the difference between analytic geometry and coordinate geometry?

Figure 5. René Descartes (by Frans Hals)

Activity 90. Label the elements of the graph in Figure 6.

1) x, y
2) I, II, III, IV
3) (0, 0)
4) (2, 3)

a) axis
b) ordered pair of point coordinates
c) origin
d) quadrant

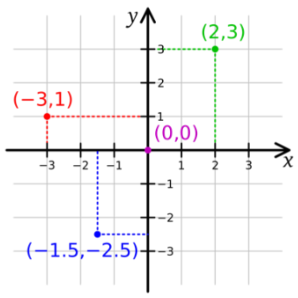


Figure 6. Coordinate Plane

Activity 91. Divide the text into paragraphs.

A straight line, usually horizontal, for which each point on the line represents a real number R is called a number line (real line). One assumes that the line extends indefinitely both to the left and to the right. A single point O on the line, called the origin, corresponds to the number zero in the real number system, and it is conventional to assume that a point “a” distance “a” units to the right of O represents the positive real number “a” and a point “b” units to the left of O the negative real number –b. The integers are thus represented as evenly spaced points, one unit apart, along the line. A number line is a one-dimensional Cartesian coordinate system (rectangular coordinate system, orthogonal coordinate system). The theory of cardinality shows that there are just as many points on the number line as there are points in a two-dimensional plane. The diagonal argument shows that the set of rational numbers (fractions) take up absolutely no space on the number line. A set of numbers used to locate a point on a number line, in a plane, or in space are called the coordinates of that point. For example, the coordinates of points on a number line could be given by their distances from a fixed point O, with points on one specified side of O being deemed a positive distance from O, and the points on the opposite side of O a negative distance from O. One way of assigning coordinates to points in the plane is to establish a fixed point O in the plane, and two lines of reference (called axes) that pass through O. Each axis is divided into a positive side and a negative side by O. Given a point P in the plane, one draws lines through P parallel to each of the axes. The distances along which these new lines intersect the axes specify the location of the point P. When the axes are drawn at right angles, the system is called a Cartesian coordinate system. The axes are usually called the x- and y-axes, and the pair of numbers (x,y) specifying the location of a point P (as “x” units along one axis, and “y” units along the second) are called the Cartesian coordinates of P. In three-dimensional space, the location of points can be specified via three mutually perpendicular (or oblique) axes passing through a common point O. The idea of assigning sets of numbers to points to specify locations is an old one. By the 3rd century B.C.E., Greek scholars Apollonius of Perga and Archimedes of Syracuse had used longitude, latitude, and altitude to define the position of a point on the Earth’s surface. Roman and Greek surveyors labelled maps with grid lines, so as to specify locations via row and column numbers. One of the biggest breakthroughs in the development of mathematics occurred when geometry and algebra were united through the invention of the Cartesian coordinate system. Credited to 17th-century French mathematician and philosopher René Descartes (whose name Latinized reads Cartesius), Cartesian coordinates (rectangular coordinates, orthogonal coordinates) provide a means of representing each point in the plane via a pair of numbers. One begins by selecting a fixed point O in the plane, called the origin, and drawing through it two perpendicular number lines, called axes, one horizontal and one vertical, and both with the point O at the zero position on the line. It has become the convention to set the positive side of the horizontal number line to the right of O, and the positive side of the vertical number line above O, and to call the horizontal axis the x-axis, and the vertical one the y-axis. The Cartesian coordinates of a point P in the plane is a pair of numbers (x,y) which then describes the location of that point as follows: *The x-coordinate, or “abscissa,” is the horizontal distance of the point from O along the horizontal axis. (A positive distance represents a point to the right of the vertical axis; a negative distance one to the left.) The y-coordinate, or “ordinate,” is the vertical distance of the point from O along the vertical axis. (A positive distance represents a point located above the horizontal axis, and a negative distance one located below.)* Extending this idea to three-dimensions, points in space can be specified by a triple of numbers (x,y,z) representing the distances along three mutually perpendicular number lines. The coordinate axes are then called the x-, y-, and z-axes. They intersect at a point O, which is zero on all three number lines. The axes could be oriented to either form a left-handed or a right-handed system. The advent of a coordinate system allowed mathematicians, for the first time, to bring the power of algebra to the study of geometry. The French mathematician Nicole Oresme (1323–82) was the first to describe a way of graphing the relationship between an independent variable and a dependent one, and thus the first to make steps toward uniting geometry and algebra. The explicit construction of what we would call a coordinate system first appeared with the work of the French lawyer and amateur mathematician Pierre de Fermat (1601–65). Starting with some horizontal reference line to represent an independent variable “x”, Fermat would graphically depict the relationship of a second variable “y” to it as a line segment, held at a fixed angle to the reference line, whose length would vary according to the variable “y” as it slides along the x-axis. Fermat did not think in terms, however, of identifying a second axis, nor did he require the line segment representing “y” to be perpendicular to the x-axis. In his famous 1637 text “Geometry”, René Descartes independently described similar methods for representing algebraic relationships graphically. Because the work of Fermat was not published until after his death, the discovery of coordinate geometry was attributed to Descartes. Because Fermat and Descartes interpreted the unknown variable “y” in an algebraic relationship as a physical length, both scholars only ever considered positive coordinates. The English mathematician John Wallis (1616–1703) was the first to introduce the possibility of negative coordinates. The idea of setting a fixed second axis, the y-axis, perpendicular to the x-axis was not popular until the mid 1700s. It was an idea that seemed to evolve gradually.

(from Elementary Geometry for College Students)

Activity 92. Based on Activity 91, for each paragraph, identify the topic sentence and provide a heading.

Activity 93. In groups, discuss the points. Refer to the text in Activity 91.

1. How did the concept of assigning coordinates to points evolve over time, from the number line to the Cartesian coordinate system? Discuss the contributions of Greek scholars, the role of maps, and the breakthroughs by mathematicians like Descartes.
2. Explore the impact of the Cartesian coordinate system on the development of mathematics. Discuss how the union of geometry and algebra changed the way mathematicians approached problem-solving and representation.
3. Compare the ancient methods of using longitude, latitude, and altitude to define a point's position with the Cartesian coordinate system. How did the understanding of coordinates in ancient times differ from the modern approach?
4. Examine the introduction of negative coordinates by John Wallis and the initial reluctance to consider them. Discuss why the concept of negative coordinates was ground-breaking and how it expanded the scope of coordinate geometry.
5. Explore the conventions established in the Cartesian coordinate system, such as positioning the positive side of the horizontal number line to the right and the positive side of the vertical number line above. Discuss the reasons behind these conventions and their importance in standardizing notation.

Activity 94. Single out the keywords (key phrases) from the text in Activity 91 and write an abstract of it.

Unit 12. Euclidean Geometry

Activity 95. Match the words with the definitions.

1. point
2. line
3. plane
4. angle
5. figure

a. a flat surface in which a straight line joining any two of its points lies entirely on that surface
b. a geometric element having no dimensions and whose position in space is located by means of its coordinates
c. any combination of points, lines, curves, or planes; a surface or space bounded on all sides by lines or planes
d. any straight one-dimensional geometrical element whose identity is determined by two points
e. the space between two straight lines that diverge from a common point or between two planes that extend from a common line

Activity 96. Classify the concepts under the headings. Represent the concepts of one class graphically.

area / broken / congruence / corner / curve / diagonal / edge / endpoint / face / height / horizontal / midpoint / parallel / perimeter / perpendicular / polygon / polyhedron / ray / segment / side / similarity / transversal / vertex / vertical / volume			
Line	**Position**	**Figure**	**Measure**

Activity 97. In pairs, discuss the questions.

1. Who is known as the “father of geometry”?
2. What is “The Elements”?
3. What is meant by Euclidean geometry?
4. Is all geometry Euclidean?

Activity 98. Read the article to examine the enduring impact of Euclid's "The Elements" on logic, mathematics, and education.

The branch of mathematics concerned with the properties of space and of figures, lines, curves, and points drawn in space is called geometry. Plane geometry examines objects drawn in a plane (lines, circles, polygons, and the like), solid geometry, or stereometry, deals with figures in three-dimensional space (polyhedra, lines, planes, and surfaces), and spherical geometry studies the properties of lines and shapes drawn on the surface of a sphere. The word "geometry" comes from the Greek words "ge" meaning "earth" and "metria" meaning "measure." As the origin of the word suggests, the study of geometry evolved from very practical concerns with regard to the accurate measurement of tracts of land, navigation, and architecture.

The geometry based on the definitions and axioms set out in Euclid's famous work "The Elements" is called Euclidean geometry. The salient feature of this geometry is that the fifth postulate, the parallel postulate, holds. It follows from this that through any point in the plane there is precisely one line through that point parallel to any given direction, that all angles in a triangle sum to precisely 180°, and that the ratio of the circumference of any circle to its diameter is always the same value π. Two-dimensional Euclidean geometry is called plane geometry, and the three-dimensional Euclidean geometry is called solid geometry. In 1899 German mathematician David Hilbert (1862–1943) proved that the theory of Euclidean geometry is free from contradiction.

Euclid of Alexandria (ca. 300–260 B.C.E.) began his famous 13-volume piece "The Elements" with 23 definitions ("a point is that which has no part" and "a line is that which has no breadth") followed by 10 axioms divided into two types: five common notions and five postulates.

His common notions were:

1. *Things that are equal to the same thing are equal to one another.*
2. *If equal things are added to equals, then the wholes are equal.*
3. *If equal things are subtracted from equals, then the remainders are equal.*
4. *Things that coincide with one another are equal to one another.*
5. *The whole is greater than the part.*

Euclid's postulates were:

1. *A straight line can be drawn to join any two points.*
2. *Any straight line segment can be extended to a straight line of any length.*
3. *Given any straight line segment, it is possible to draw a circle with centre one endpoint and with the straight line segment as the radius.*
4. *All right angles are equal to one another.*

5. *If two straight lines emanating from the endpoints of a given line segment have interior angles on one given side of the line segment summing to less than two right angles, then the two lines, if extended, meet to form a triangle on that side of the line segment.*

It is worth noting that Euclid deliberately avoided any direct mention of the notion of infinity. His wording of the second postulate, for instance, avoids the need to state that straight lines can be extended indefinitely, and his fifth postulate, also known as the parallel postulate, avoids direct mention of parallel lines, that is, lines that never meet when extended indefinitely.

From these basic assumptions Euclid deduced, by pure logical reasoning, 465 statements of truth (theorems) about geometric figures. The systematic approach he followed and the rigour of reasoning he introduced was hailed as a great intellectual achievement. His model of mathematical exploration became the standard for all mathematical research for the next 2,000 years.

Euclid's fifth postulate was always regarded with suspicion. It was never viewed as simple and as self-evident as his remaining four postulates, and Euclid himself did his utmost to avoid using it in his work. (Euclid did not invoke the fifth postulate until his 29th proposition.) Over the centuries scholars came to believe that the fifth postulate could be logically deduced from the remaining four postulates and therefore did not need to be listed as an axiom. Many people proposed proofs for it, including the 5th-century Greek philosopher Proclus, who is noted for his historical account of Greek geometry. Unfortunately, his proof was flawed, as were the proofs proposed by Arab scholars of the 8th and 9th centuries, and by Western scholars of the Renaissance.

In 1733 Italian teacher and scholar Girolamo Saccheri (1667–1733) believed that because Euclid's axioms model the real world, which he thought to be consistent, they cannot lead to a contradiction. If the first four postulates do indeed imply that the fifth postulate is also true, then assuming the four postulates together with the negation of the fifth postulate should lead to a logical inconsistency. Unfortunately, in following this tact, Saccheri never came across a contradiction.

In 1795 Scottish mathematician and physicist John Playfair (1748–1819) proposed an alternative formulation of the famous fifth postulate (today known as Playfair's axiom). It states:

In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.

This version of the axiom is considerably easier to handle, and its negation is easier to envision. In an attempt to follow Saccheri's approach, Russian mathematician Nikolai Ivanovich Lobachevsky (1792–1856) and Hungarian mathematician János Bolyai (1802–1860), independently came to the same surprising conclusion: the first four of Euclid's postulates together with the negation of Playfair's version of the fifth postulate will not lead to a contradiction. This established, once and for all, that the fifth postulate is an independent axiom and cannot be deduced from the remaining four postulates. More important, by exploring the geometries that result in assuming that the fifth postulate does not hold, scholars were led to the discovery of non-Euclidean geometry.

In the late 1800s the German mathematician David Hilbert (1862–1943) noted that, despite its rigour, Euclid's work contained many hidden assumptions. He also realized, despite Euclid's attempts to describe them, that the notions of "point," "line," and "plane" cannot be properly defined and must remain as undefined terms in any theory of geometry. In his 1899 work "Foundations of Geometry" Hilbert refined and expanded Euclid's postulates into a list of 28 basic assumptions that define all that is needed in a complete account of Euclid's geometry. His axioms are today referred to as Hilbert's axioms.

(from Elementary Geometry for College Students)

Table 28. Polygons

Polygon	Edges Vertices	Polygon	Edges Vertices	Polygon	Edges Vertices
		undecagon hendecagon	11		
		dodecagon	12	icosagon	20
triangle trigon	3	tridecagon triskaidecagon	13	triacontagon	30
quadrilateral tetragon	4	tetradecagon tetrakaidecagon	14	tetracontagon	40
pentagon	5	pentadecagon pentakaidecagon	15	pentacontagon	50
hexagon	6	hexadecagon hexakaidecagon	16	hexacontagon	60
heptagon	7	heptadecagon heptakaidecagon	17	heptacontagon	70
octagon	8	octadecagon octakaidecagon	18	octacontagon	80
nonagon enneagon	9	enneadecagon enneakaidecagon	19	enneacontagon	90
decagon	10			hectogon	100

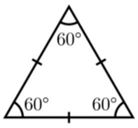
Activity 99. Match the two parts of the sentences to classify angles.

1. An angle of 0°	a. is called acute.
2. An angle between 0° and 90°	b. is called obtuse.
3. An angle of 90°	c. is called a reflex angle.
4. An angle between 90° and 180°	d. is called a round angle (or a perigon).
5. An angle of 180°	e. is called a straight angle.
6. An angle between 180° and 360°	f. is called a right angle.
7. An angle of 360°	g. is called a null angle.

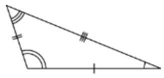
Activity 100. Label the triangles (trigons). Describe their geometric properties.

Lengths of Sides

1) a scalene triangle
2) an equilateral triangle
3) an isosceles triangle

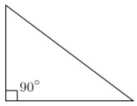
a) 

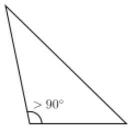
b) 

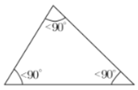
c) 

Internal Angles

4) a right triangle
   a right-angled triangle
5) an acute triangle
   an acute-angled triangle
6) an obtuse triangle
   an obtuse-angled triangle

d) 

e) 

f) 

Activity 101. Who was the first to prove Pythagoras' theorem? Watch the video "How Many Ways Are There to Prove the Pythagorean Theorem?" to choose the best answer to the questions. Then watch the video again and make a note of the famous proofs in the video. Offer more proofs.

https://disk.yandex.ru/i/bW-1-zx61JsT9g

1. What do Euclid, young Einstein, and President Garfield have in common?
A. They all taught mathematics at universities
B. They all created proofs for the Pythagorean theorem
C. They all discovered new geometric shapes
D. They all studied ancient Babylonian mathematics

2. How did ancient Egyptian surveyors likely use the Pythagorean theorem in their work?
A. They used it to measure the height of pyramids
B. They used a knotted rope to create right angles for building
C. They calculated distances between cities
D. They designed irrigation systems for farms

3. What is the main purpose of mathematical proofs?
A. To make mathematics more difficult for students
B. To show that a theorem works for all cases, not just specific examples
C. To honor famous mathematicians from history
D. To create new mathematical rules and formulas

4. In the proof by rearrangement, what stays the same when the triangles are moved?
A. The shape of the overall figure and the area of the triangles
B. The angles of the triangles and the length of the sides
C. The total area of the figure and the area of the triangles
D. The position of the squares and the size of the hypotenuse

5. What can be inferred about the Pythagorean theorem?
A. It only works for triangles with specific measurements
B. It has inspired many different ways of proving the same mathematical truth
C. It was forgotten after Pythagoras died and rediscovered later
D. It is too complicated for most people to understand

Unit 13. Non-Euclidean Geometries

Activity 102. Label each geometric figure as flat (two-dimensional) or solid (three-dimensional). Add more figures of each type.

1. ball
2. circle
3. cone
4. cube
5. cylinder
6. disk
7. parallelepiped
8. parallelogram
9. polygon
10. polyhedron
11. prism
12. pyramid
13. quadrilateral
14. rectangle
15. rhombus
16. sphere
17. square
18. trapezium
19. trapezoid
20. triangle

Activity 103. Explain the concepts graphically in relation to the circle.

1. arc
2. centre
3. chord
4. circumference
5. diameter
6. origin
7. pi
8. quadrant
9. radius
10. secant
11. sector
12. segment
13. semicircle
14. tangent

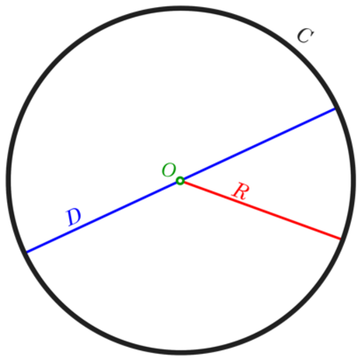


Figure 7. Circle

Activity 104. What are the three types of conic sections? Watch the video "The Mathematics of Sidewalk Illusions" to choose the best answer to the questions. Offer a geometric interpretation of illusory street art.

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1. What is anamorphosis?
A. A painting technique that uses only bright colors
B. A special type of perspective art that creates realistic 3D views on 2D surfaces
C. An ancient Roman method of drawing small figures

D. A mathematical formula used in modern computer graphics

2. What must a viewer do to see an anamorphic image correctly?
A. Look at it from multiple angles at the same time
B. Stand very close to the artwork with both eyes open
C. Position themselves in exactly the right spot
D. Use special glasses to see the hidden image

3. How did Leonardo da Vinci create the first known anamorphic drawing?
A. He painted directly onto a window at an angle
B. He used mathematical principles to manipulate perspective
C. He copied an ancient Greek technique
D. He projected light through colored glass

4. Why do parallel lines in perspective drawings need to converge to a vanishing point?
A. They must always be drawn at an angle to look realistic
B. All lines in art should meet at the center
C. They are only drawn parallel if they're parallel to the canvas plane
D. Renaissance artists preferred this artistic style

5. What is the main purpose of creating anamorphic sidewalk drawings?
A. To teach people about Renaissance art techniques
B. To create the illusion that a 3D image has been added seamlessly into an existing scene
C. To make exact copies of famous paintings on the ground
D. To show that sidewalks can be used as art galleries

Activity 105. Read the article to complete the table differentiating between the three geometries.

After numerous unsuccessful attempts throughout history to establish the parallel postulate as a consequence of the remaining four of Euclid's postulates, mathematicians began to contemplate theories of geometry in which the fifth postulate does not hold. Any such theory of geometry is called a non-Euclidean geometry.

In 1795 the Scottish mathematician and physicist John Playfair (1748–1819) presented an alternative, but equivalent, formulation of the parallel postulate:

Through any point in the plane, there is precisely one line through that point parallel to any prescribed direction.

Recasting the postulate this way makes it apparent that negation of the famous fifth postulate has two parts. Either:

1. *There are no lines through a given point parallel to a given direction.*
2. *There is more than one line through a given point parallel to a given direction.*

Independently discovered in 1823 by the Hungarian mathematician János Bolyai (1802–60) and in 1829 by the Russian mathematician Nikolai Ivanovich Lobachevsky (1792–1856), hyperbolic geometry (Lobachevskian geometry) is a non-Euclidean geometry in which the famous parallel postulate fails in the following manner:

Through a given point not on a given line, there is more than one line parallel to that given line.

The French mathematician Jules-Henri Poincaré (1854–1912) later provided a simple model for this geometry and the means to easily visualize geometric results in this theory. The "Poincaré disk" consists of all the points in the interior of the unit circle. A "point" in hyperbolic geometry is any point inside this circle, and a "line" is to be interpreted as a circular arc within the circle with endpoints perpendicular to the boundary of the circle. Any diameter of the boundary circle is also considered a line. Distances are not measured with a traditional ruler: points on the boundary circle are considered to be infinitely far from the centre of the circle.

Bolyai and Lobachevsky showed that all but the fifth of Euclid's postulates hold in the hyperbolic geometry and, moreover, that this model of geometry is consistent (that is, free of contradictions). This establishes that the parallel postulate cannot be logically deduced as a consequence of the remaining axioms proposed by Euclid.

In hyperbolic geometry, all angles in triangles sum to less than 180°, and the ratio of the circumference of any circle to its diameter is less than π. (Moreover, the value of this ratio is not the same for all circles.) Also, it is possible for two perpendicular lines to be parallel to the same line.

Physicists, following the work of Albert Einstein, suggest that the geometry of our universe is hyperbolic: that it appears to us as Euclidean is a result of the fact that we occupy such a small portion of it. (This is analogous to the fact that it is difficult to recognize the Earth as round when living on it.)

Discovered in 1856 by the German mathematician Georg Friedrich Bernhard Riemann (1826–66), and later slightly modified by Felix Klein (1849–1925), elliptic geometry (Riemannian geometry) is a non-Euclidean geometry in which the famous parallel postulate fails in the following manner:

Through a given point not on a given line, there are no lines parallel to that given line.

Riemann used the surface of a sphere as a model of this geometry by interpreting the word "line" to mean a great circle on the sphere. Given that in a theory of geometry two lines are meant to intersect at just one point (yet any two great circles intersect at two antipodal points), it is appropriate then to interpret the word "point" in elliptic geometry as an antipodal pair of points on the surface. In this setting, it is now also true that any two distinct points determine a unique line.

Riemann and Klein proved that all but the fifth of Euclid's postulates hold in this model and, moreover, that this model is consistent (that is, free of contradictions). This establishes that the parallel postulate cannot be logically deduced as a consequence of the remaining axioms proposed by Euclid.

In elliptic geometry all angles in triangles sum to more than 180°, and the ratio of the circumference of any circle to its diameter is greater than π (and this value varies from circle to circle).

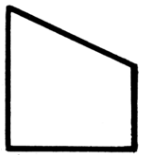
(from Elementary Geometry for College Students)

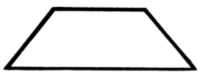
№	Geometry	Discoverer	Postulates	Properties
1	hyperbolic			
2	Euclidean			
3	elliptic			

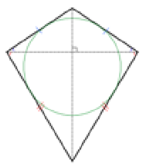
Activity 106. Label the quadrilaterals (tetragons). Describe their geometric properties.

1) a kite, a deltoid
2) an irregular quadrilateral (BrE)
   a trapezium (AmE)
3) a trapezium (BrE)
   a trapezoid (AmE)
4) an isosceles trapezium (BrE)
   an isosceles trapezoid (AmE)

a) 

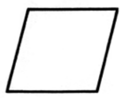
b) 

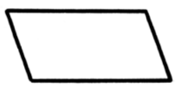
c) 

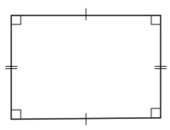
d) 

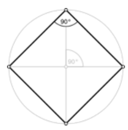
Parallelograms

5) a rectangle
6) a rhomboid
7) a rhombus, a rhomb, a diamond
8) a square

e) 

f) 

g) 

h) 

Activity 107. Reorder the sentences to make a text on topology.

a. All that is required of a topological transformation is that points that begin close together remain relatively close together.
b. Any transformation that requires puncturing or tearing a surface, or joining together two disjoint portions of a figure, is not considered a valid topological action. (Tearing a surface, for instance, separates points that were close together.)
c. As it is impossible to deform a sphere into a torus without puncturing the surface, these two shapes are not topologically equivalent.
d. As other examples, the capital letters C, M, and Z are topologically equivalent, as are the letters D, O, P, and R, but no letter from the first group is topologically equivalent to a letter in the second.
e. For example, the statement, "Removing a point from a circle produces a line segment" is valid for the entire class of objects topologically equivalent to a circle.
f. In 1895 French mathematician Henri Poincaré (1854–1912) examined these works and published five papers, laying a theoretical framework for a formal study of topology.
g. In this viewpoint, a circle and a square (of any size) are topologically equivalent, since either can be continuously deformed into the other, and a number of geometrical properties apply equally well to either object.
h. No letter in the alphabet is topologically equivalent to the letter B (other than capital B itself).
i. The mathematician Carl Friedrich Gauss (1777–1855) examined the distortion of knots and invariant properties that arise in the study of projective geometry.
j. The study of those properties of geometric figures and surfaces that remain unchanged when the shapes of those objects are distorted by a continuous deformation (such as stretching, shrinking, or twisting) is called topology.
k. The study of topology began with Leonhard Euler (1707–83) and his analysis of the famous seven bridges of Königsberg problem.
l. Unlike a classical geometer, a topologist is not interested in questions of distances and angles but is only concerned with the relative positions of points.

Activity 108. Choose one geometric figure, flat or solid, to present. Demonstrate a method of its construction as well as a description of its geometric properties.

Unit 14. Development of Geometry

Activity 109. Complete the sentences with the opposites of the words in brackets by adding negative prefixes.

1. ___________ (Euclidean) geometry explores spaces with different postulates than classical geometry.
2. ___________ (numeracy) refers to a lack of basic numerical skills.
3. ___________ (rational) numbers cannot be expressed as fractions of integers.
4. Be cautious not to ___________ (calculate) the values in your test.
5. Despite the tutorial, some students remain ___________ (numerate) in basic calculations.
6. One is an ___________ (divisible) number in the realm of integers.
7. Solve the ___________ (equality) to determine the valid range of values.
8. The ___________ (calculation) in the initial step led to an incorrect result.
9. The ___________ (proper) fraction needed to be converted into a mixed number.
10. The complex ___________ (equation) involves variables on both sides of the equation.
11. The event is statistically ___________ (probable) given the current conditions.
12. The formula provided an ___________ (correct) solution to the mathematical problem.
13. The function exhibits ___________ (bounded) growth as x approaches infinity.
14. The lengths of the two lines are ___________ (equal) in this geometric figure.
15. The mathematical relationship remains ___________ (variable) under certain conditions.
16. The measurement was ___________ (exact) due to the limitations of the measuring tool.
17. The possibilities are ___________ (limited) when dealing with infinite sets.
18. The shape displays an ___________ (symmetrical) arrangement of elements.
19. The value of the variable remains ___________ (known) in this equation.

Activity 110. Do the quiz on the advancement of geometry. In pairs, compare your answers.

1. **When did Egyptian and Babylonian scholars develop principles of measurement and spatial reasoning?**
   a. 1900 B.C.E.
   b. 1650 B.C.E.
   c. 300–260 B.C.E.
   d. the 17^{th} century
2. **What method did Egyptian scholars use to construct right angles?**
   a. algebraic equations
   b. the Pythagorean theorem
   c. knotted ropes
   d. trigonometric functions
3. **Who compiled a large volume of geometric knowledge in "The Elements" and introduced logical reasoning in geometry?**
   a. Isaac Newton
   b. Euclid
   c. René Descartes
   d. Carl Friedrich Gauss
4. **What breakthrough in geometry occurred in the 17^{th} century with the work of René Descartes?**
   a. introduction of calculus
   b. discovery of non-Euclidean geometry
   c. use of Cartesian coordinates
   d. development of hyperbolic geometry
5. **Who pursued the task of developing a full algebraic model of geometry after Descartes?**
   a. Carl Friedrich Gauss
   b. Euclid
   c. John Playfair
   d. Pierre de Fermat
6. **Which mathematician inspired work on the theory of differential calculus through his contributions to geometry?**
   a. Albert Einstein
   b. Carl Friedrich Gauss
   c. Euclid

d. Pierre de Fermat

7. **Who first permitted negative values for distances in coordinate geometry?**
   a. Sir Isaac Newton
   b. Gottfried Wilhelm Leibniz
   c. René Descartes
   d. John Playfair
8. **What did the Scottish mathematician John Playfair propose regarding Euclid's fifth postulate?**
   a. eliminating the fifth postulate
   b. proving the fifth postulate as a consequence of the remaining four
   c. introducing an equivalent postulate
   d. revising the fifth postulate
9. **What type of geometry did Nikolai Ivanovich Lobachevsky develop, challenging Euclid's fifth postulate?**
   a. Euclidean geometry
   b. hyperbolic geometry
   c. spherical geometry
   d. elliptic geometry
10. **What ground-breaking idea did Bernhard Riemann propose in his 1854 lecture?**
   a. the existence of parallel lines
   b. the study of dimensions
   c. non-Euclidean geometry
   d. the foundations of calculus

Activity 111. Read the article. Review your answers to the quiz in Activity 110.

The study of geometry is an ancient one. Records show that Egyptian and Babylonian scholars of around 1900 B.C.E. had developed sound principles of measurement and spatial reasoning in their architecture and in their surveying of land. Both cultures were aware of Pythagoras's theorem and had developed tables of Pythagorean triples. (The Egyptians used knotted ropes to construct "3-4-5 triangles" to create right angles.) Ancient Indian texts on altar construction and temple building demonstrate sophisticated geometry knowledge, and the famous volume "The Nine Chapters on the Mathematical Art" from ancient China also includes work on the Pythagorean theorem.

In ancient Greece, mathematical scholars came to realize that many properties of shapes and figures could be deduced logically from other properties. In his epic work "The Elements" the Greek geometer Euclid (ca. 300–260 B.C.E.) collated a large volume of knowledge on the subject and showed that each and every result could be logically deduced from a very small set of basic assumptions (self-evident truths) about how geometry should work. Euclid's work was rigorous and systematic, and the notion of a logical proof was born. Euclid's postulates and the process of logical reasoning became the model of all further geometric investigation for the two millennia that followed. His method of compiling and organizing all mathematical knowledge known at his time was a significant intellectual achievement. Euclid's rigorous approach was, and still is, modelled in other branches of mathematics. Scholars in set theory, the foundations of mathematics, and calculus, for instance, all seek to follow the same process of formal reasoning as the correct approach to achieve proper understanding of these topics.

The next greatest breakthrough in the advancement of geometry occurred in the 17th century with the discovery of Cartesian coordinates as a means to represent points as pairs of real numbers and lines and curves as algebraic equations. This approach, described by the French mathematician and philosopher René Descartes (1596–1650) in his famous 1637 work "Geometry", united the then-disparate fields of algebra and geometry. Unfortunately, Descartes's interests lay only in advancing methods of geometric construction, not in developing a full algebraic model of geometry. This latter task was pursued by the French mathematician Pierre de Fermat (1601–65), who had also outlined the principles of coordinate geometry in an unpublished manuscript that he had circulated among mathematicians before the release of "Geometry". Fermat later published the work in 1679 under the title "On the Plane and Solid Locus". The application of algebra to the discipline provided scholars a powerful new tool for solving geometric problems, and also provided them with a large number of different types of curves for study.

Fermat's work in geometry inspired work on the theory of differential calculus and, later, led to the study of "differential geometry" (the application of calculus to the study of shapes and surfaces). This was developed by the German mathematician and physicist Carl Friedrich Gauss (1777–1855).

Neither Descartes nor Fermat permitted negative values for distances. Consequently, neither scholar worked with a full set of coordinate axes as we use them today. The notions of negative distance and negative area were first put forward by Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716), the coinventors of calculus.

The 19th century saw other major advances in geometry. It had long been noted that Euclid's fifth postulate, the so-called parallel postulate, is not necessary for a great deal of geometry. Many Arab scholars of the first millennium attempted, without success, to show that the fifth postulate could be logically deduced from the remaining four (thereby rendering it unnecessary), as did European scholars of the Renaissance. In 1795 the Scottish mathematician John Playfair showed that the fifth postulate is equivalent to the statement that, through any point, one can draw one, and only one, line through that point parallel to a given line. (This is today called Playfair's axiom.) Although not eliminating the need for the fifth postulate, Playfair showed that it could be understood in a more tractable form.

In 1829 the Russian mathematician Nikolai Ivanovich Lobachevsky (1792–1856) took a bold step and considered a geometric world in which the fifth postulate is false. He assumed that through a given point more than one line could be drawn parallel to a given line. In doing this, Lobachevsky discovered a new, consistent mathematical system free from contradiction, one as logically valid as the geometry of Euclid. (This geometry is today called hyperbolic geometry.) The philosophical impact of Lobachevsky's work was enormous: he had shown that mathematics need not be based on a single set of physical truths, and that other equally valid mathematical systems do exist based on alternative, carefully chosen axioms. Lobachevsky had also shown that Euclid's fifth postulate cannot be established as a consequence of the remaining four axioms: he had presented a valid example of a system in which the first four of Euclid's postulates hold, but the fifth does not.

Surprisingly some of Lobachevsky's ideas were anticipated well before the 19^{th} century. The great Persian mathematician and poet Omar Khayyam (ca. 1048–1122) established a number of results that we recognize today as non-Euclidean. These results were later translated into Latin, and extended upon, by the Italian priest Girolamo Saccheri (1667–1733). Unfortunately, neither scholar discovered the validity of non-Euclidean geometry, as each was focused instead on trying to establish Euclid's fifth postulate as a consequence of the remaining four.

The German mathematician Bernhard Riemann (1826–66) discovered an alternative form of non-Euclidean geometry in which Euclid's fifth postulate fails in a different way. In a system of spherical geometry, it is never possible to draw a line through a given point parallel to a given line.

Riemann's contributions to the advancement of geometry were significant. In his famous 1854 lecture "On the Hypotheses that Lie at the Foundation of Geometry", Riemann put forward the view that geometry can be the study of any kind of space of any number of dimensions, and later developed the mathematics needed to properly describe the shape of space. Albert Einstein (1879–1955) later used this work to develop his theory of relativity.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 112. Identify the individuals based on the descriptions from the text in Activity 111.

Albert Einstein / Bernhard Riemann / Carl Friedrich Gauss / Euclid / Girolamo Saccheri / Gottfried Wilhelm Leibniz / John Playfair / Nikolai Ivanovich Lobachevsky / Omar Khayyam / Pierre de Fermat / René Descartes / Sir Isaac Newton

1. The coinventor of calculus, who first permitted negative values for distances and introduced the notions of negative distance and negative area.
2. A French mathematician and philosopher who introduced Cartesian coordinates in his work "Geometry," uniting algebra and geometry.
3. A French mathematician who outlined the principles of coordinate geometry and applied algebra to solve geometric problems.
4. A German mathematician and physicist who developed "differential geometry," applying calculus to the study of shapes and surfaces.
5. A German mathematician who discovered an alternative form of non-Euclidean geometry (spherical geometry) and proposed that geometry can be the study of any space of any number of dimensions.
6. A Greek geometer who compiled "The Elements," a comprehensive work on geometry, and established logical proofs as a model for geometric investigations.
7. An Italian priest who translated and extended upon Omar Khayyam's results, focused on trying to establish Euclid’s fifth postulate as a consequence of the remaining four.
8. A Persian mathematician and poet who established results recognized today as non-Euclidean, although he did not discover the validity of non-Euclidean geometry.
9. A physicist who used Riemann's work to develop his theory of relativity.
10. A Russian mathematician who developed hyperbolic geometry by considering a geometric world in which Euclid’s fifth postulate is false.
11. A Scottish mathematician who, in 1795, showed the equivalence of the parallel postulate to the statement that through any point, one can draw one, and only one, line through that point parallel to a given line (now called Playfair’s axiom).

Activity 113. Rearrange the events in chronological order according to the text of Activity 111. Provide dates where possible.

a. Albert Einstein's use of Riemann's work in the development of the theory of relativity.
b. Bernhard Riemann's discovery of an alternative form of non-Euclidean geometry.

c. Carl Friedrich Gauss's contributions to the theory of differential calculus and differential geometry.
d. Development of "The Nine Chapters on the Mathematical Art" in ancient China.
e. Development of hyperbolic geometry by Nikolai Ivanovich Lobachevsky.
f. Discovery of Cartesian coordinates and representation of points as pairs of real numbers.
g. Introduction of negative values for distances by Sir Isaac Newton and Gottfried Wilhelm Leibniz.
h. Invention of algebraic methods for geometric construction by René Descartes.
i. John Playfair's demonstration of the equivalence of Euclid's fifth postulate to Playfair's axiom.
j. Omar Khayyam's establishment of results recognized today as non-Euclidean.
k. Publication of Euclid's "The Elements" and the introduction of logical reasoning in geometry.
l. Recognition of Pythagoras's theorem and the use of knotted ropes to create "3-4-5 triangles" by Egyptian scholars.
m. Pierre de Fermat's publication of "On the Plane and Solid Locus" and principles of coordinate geometry.
n. John Playfair's work on the parallel postulate.

Activity 114. Determine whether the statements are true or false by quoting from the text in Activity 111.

1. A significant advancement in geometry during the 17th century was the introduction of Cartesian coordinates by René Descartes (1596–1650), which fused algebra and geometry and was further developed by Pierre de Fermat (1601–65).
2. Bernhard Riemann (1826–66) advanced non-Euclidean geometry by incorporating hyperbolic geometry, proposing that geometry could explore spaces of various dimensions, a concept later utilized by Albert Einstein in his theory of relativity.
3. Euclid, the Greek geometer from approximately 300 to 260 B.C.E., created "The Elements," emphasizing logical deductions from basic principles as the cornerstone of geometry for centuries.
4. Geometry only gained prominence in the 19th century, marked by breakthroughs like Cartesian coordinates and hyperbolic geometry.
5. Girolamo Saccheri (1667–1733) recognized non-Euclidean geometry's validity while attempting to prove Euclid's fifth postulate.
6. John Playfair's axiom successfully removed the necessity of Euclid's fifth postulate, streamlining the principles of geometry.
7. Newton and Leibniz advocated for negative values in distances, reshaping the use of coordinate axes.

8. Nikolai Ivanovich Lobachevsky (1792–1856) introduced hyperbolic geometry by questioning Euclid's fifth postulate, establishing an internally consistent mathematical system distinct from Euclidean geometry.
9. René Descartes (1596–1650) prioritized creating a comprehensive algebraic model for geometry over enhancing methods of geometric construction.
10. The exploration of geometry has ancient origins, with Egyptian and Babylonian scholars around 1900 B.C.E. establishing fundamental principles of measurement and spatial reasoning.

Activity 115. In groups, discuss the points. Refer to the text in Activity 111.

1. Explore the insights into ancient civilizations, such as Egypt, Babylon, India, and China, and their sophisticated understanding of geometry principles, including Pythagoras's theorem.
2. Discuss the significant impact of Euclid's work in "The Elements," examining how his logical approach and postulates shaped geometric investigation for over two millennia.
3. Delve into the 17^{th}-century breakthrough of Cartesian coordinates, as described by Descartes, and its role in uniting algebra and geometry. Consider the subsequent developments by Fermat.
4. Examine the shift from geometric construction to an algebraic model of geometry initiated by Fermat and its application in solving geometric problems and studying various curves.
5. Explore how Fermat's work influenced the development of differential calculus and led to the study of differential geometry by Carl Friedrich Gauss in the 18^{th} century.
6. Discuss the introduction of negative values for distances by Newton and Leibniz, and how this departure from Descartes and Fermat influenced coordinate axes and geometric understanding.
7. Investigate the historical attempts by Arab and European scholars to logically deduce Euclid's fifth postulate, leading to Playfair's axiom and the recognition of alternative interpretations.
8. Explore Lobachevsky's bold departure from Euclidean geometry, introducing hyperbolic geometry and the philosophical impact of showing that mathematics can be based on different axioms.
9. Discuss how Omar Khayyam and Girolamo Saccheri anticipated non-Euclidean ideas, even though they did not fully recognize the validity of such geometry.
10. Investigate Riemann's contributions to spherical geometry and the subsequent use of his work by Albert Einstein in developing the theory of relativity, illustrating the interdisciplinary nature of geometry.

Activity 116. Choose one point in Activity 115 and elaborate on it in writing. Refer to the text in Activity 111.

Unit 15. Trigonometry

Activity 117. Complete the sentences with the derivatives of the words in brackets.

1. __________ (algebra) expressions and equations involve variables and constants, allowing for the generalization of mathematical relationships.
2. __________ (analyze) geometry combines algebraic and geometric techniques to study geometric shapes using coordinate systems.
3. __________ (arithmetic) calculations are essential in everyday tasks, such as budgeting and financial planning.
4. __________ (geometry) series have a common ratio between consecutive terms.
5. __________ (logic) reasoning is a fundamental aspect of mathematical thinking, guiding the deduction of conclusions based on established principles.
6. __________ (statistics) analyzing a set of data involves calculating measures such as mean, median, and standard deviation to understand the central tendency and variability within the dataset.
7. __________ (topology) spaces provide a framework for studying properties preserved under continuous transformations.
8. __________ (trigonometry) functions, such as sine and cosine, describe relationships between angles and sides in triangles.
9. A __________ (statistics) applies mathematical and statistical techniques to analyze and interpret data.
10. A coffee cup and a donut are considered __________ (topology) equivalent.
11. A data __________ (analyze) employs mathematical and statistical techniques to interpret complex datasets and extract meaningful insights.
12. Elementary __________ (arithmetic) involves basic operations like addition, subtraction, multiplication, and division.
13. Mathematical __________ (analyze) involves the study of limits, continuity, and derivatives, essential for understanding functions and their behaviour.
14. Scientists use __________ (statistics) methods to analyze experimental data and draw meaningful conclusions.
15. The sine of an angle in a right-angled triangle is calculated __________ (trigonometry) as the ratio of the opposite side to the hypotenuse.
16. When solving a quadratic equation, one can find the roots __________ (algebra) by applying the quadratic formula.

Activity 118. Read the passage. In pairs, discuss the questions in the box.

The word "trigonometry" comes from the Greek words "trigonon" ("triangle") and "metron" ("to measure"). Contrary to its name, the theory of trigonometry (informally abbreviated to "trig") is best motivated as a theory about circles, not triangles. (This, in fact, matches the historical development of the subject.) Trigonometry is the branch of mathematics concerned with specific functions of angles and their application to calculations. There are six functions of an angle commonly used in trigonometry. Their names and abbreviations are sine (sin), cosine (cos), tangent (tan), cotangent (cot), secant (sec), and cosecant (csc). These six trigonometric functions in relation to a right triangle are displayed in the figure. For example, the triangle contains an angle A, and the ratio of the side opposite to A and the side opposite to the right angle (the hypotenuse) is called the sine of A, or sin A; the other trigonometry functions are defined similarly. These functions are properties of the angle A independent of the size of the triangle, and calculated values were tabulated for many angles before computers made trigonometry tables obsolete. Trigonometric functions are used in obtaining unknown angles and distances from known or measured angles in geometric figures. Trigonometry developed from a need to compute angles and distances in such fields as astronomy, mapmaking, surveying, and artillery range finding. Problems involving angles and distances in one plane are covered in plane trigonometry. Applications to similar problems in more than one plane of three-dimensional space are considered in spherical trigonometry.

(from Encyclopaedia Britannica)

1. What is trig?
2. Does the etymology of the word "trigonometry" reveal its meaning?
3. What are the six common trigonometric functions?
4. What is the use of the Greek letter θ (theta) in trigonometry?
5. What are the application areas of trigonometry?

Activity 119. Match the trigonometric functions with the ratios.

1. sine (sin)
2. cosine (cos)
3. tangent (tan / tg)
4. cosecant (csc / cosec)
5. secant (sec)
6. cotangent (cot / ctg)

a. $\frac{adjacent}{hypotenuse}$

b. $\frac{adjacent}{opposite}$

c. $\frac{hypotenuse}{adjacent}$

d. $\frac{hypotenuse}{opposite}$

e. $\frac{opposite}{adjacent}$

f. $\frac{opposite}{hypotenuse}$

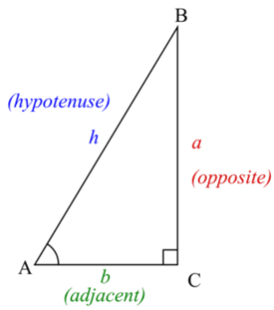


Figure 8. Right Triangle

Activity 120. What is the connection between navigation and trigonometry? Watch the video "How Does Math Guide Our Ships at Sea?" to choose the best answer to the questions Describe the function that trigonometry originally fulfilled.

https://disk.yandex.ru/i/YGDQZvzsvLow_Q

1. What was the main problem with dead reckoning as a navigation method?
A. It required too many sailors to operate the ship
B. Small errors in direction could cause ships to miss their destination
C. It only worked during daylight hours
D. It was too expensive for most ships to use

2. Why was it necessary to know the exact time back in England while at sea?
A. To calculate when the ship would arrive at its destination
B. To determine the ship's longitude by comparing sun angles
C. To know when to change the ship's direction
D. To record the ship's daily progress in the logbook

3. What was the main disadvantage of John Harrison's clock?
A. It wasn't accurate enough for navigation
B. It couldn't work in harsh sea conditions
C. It was very costly because it was handmade
D. It was too heavy to carry on most ships

4. How did Henry Briggs contribute to making logarithms more practical?
A. He invented logarithms in his castle in Scotland
B. He suggested changing the base to make calculations simpler
C. He created the first clock that could work at sea
D. He developed the sextant for measuring angles

5. What does the history of navigation demonstrate about creativity?
A. It shows that mathematics is the most important skill for innovation
B. It proves that working alone leads to the best inventions
C. It illustrates that collaboration between different fields is valuable
D. It suggests that expensive tools are necessary for progress

Activity 121. Do the quiz on trigonometry. In pairs, compare your answers.

1. **When did the use of triangles for determining distances, leading to the birth of trigonometry, begin?**
   a. the 2nd century B.C.E.
   b. the 5th century
   c. the 12th century
   d. the 17th century
2. **Which ancient Greek philosopher is credited with using similar triangles to determine the height of the Cheops pyramid?**
   a. Eratosthenes of Cyrene
   b. Hipparchus of Rhodes
   c. Thales of Miletus
   d. Claudius Ptolemy
3. **What did Greek astronomers choose to work with instead of angles when developing models for the motion of stars and planets?**
   a. sine values
   b. chords of circles
   c. tangent functions
   d. secant values
4. **In the 5th century C.E., Indian scholars simplified the theory of chords by working with:**
   a. chord values
   b. sine ratios
   c. half-chords for half-angles
   d. tangent functions
5. **Which mathematician is believed to have systemized theorems and proofs of Indian trigonometry and possibly invented the tangent function?**
   a. Fibonacci
   b. Abu al-Wafa
   c. Georg Joachim Rheticus
   d. Leonhard Euler
6. **Who coined the terms "tangent" and "secant" and developed fundamental trigonometric formulae in the 17th century?**
   a. Regiomontanus
   b. François Viète
   c. Thomas Fincke

d. Bartholomaeus Pitiscus

7. **When did the perspective of thinking about trigonometric quantities as ratios rather than actual line lengths emerge?**
    a. the 12th century
    b. the 17th century
    c. the 18th century
    d. the 19th century
8. **Which mathematician wrote "Introductio in analysin infinitorum" in 1748, outlining principles of trigonometry as we regard them today?**
    a. Regiomontanus
    b. Claudius Ptolemy
    c. François Viète
    d. Leonhard Euler

Activity 122. Read the article. Review your answers to the quiz in Activity 121.

From very early times, surveyors, architects, navigators, and astronomers have made use of triangles to determine distances that could not be measured directly. This gave birth to the subject we today know as trigonometry. There are problems in the ancient Egyptian text, the Rhind papyrus of around 1650 B.C.E., that call for the determination of the slope angles of pyramid faces using the equivalent of the cotangent function we use today, and a Babylonian clay tablet from around 1700 B.C.E. contains a table of secant values for the 15 angles between 30° and 45°. The Greek philosopher Thales of Miletus (ca. 600 B.C.E.) is said to have made use of similar triangles to determine the height of the Cheops pyramid by comparing the length of its shadow with the length of the shadow of a rod inserted in the ground. Eratosthenes of Cyrene (ca. 250 B.C.E.) computed the circumference of the Earth using lengths of shadows and a simple geometric argument on angles.

Greek astronomers of ancient times believed that the stars and planets of the night sky moved along circular arcs of a giant celestial sphere, and they worked to develop models that would accurately predict the motion of these objects on the sphere. Rather than phrase matters in terms of angles, which proved to be difficult, Greek astronomers chose to work with measures of straight lengths closely related to angles, namely, the lengths of chords of circles.

Hipparchus of Rhodes (ca. 200 B.C.E.) constructed a table of such chord lengths for a circle of circumference 21,600 = 360 × 60 units (which corresponds to 1 unit of circumference for each minute of arc).

In the 2nd century C.E., the mathematician Claudius Ptolemy wrote the first extensive treatise on the theory of chords and their use in obtaining information about "spherical triangles," that is, triangles made by great circular arcs on the surface of a sphere. In addition to working out theorems, Ptolemy explained how to construct tables of chord values, and presented his own list of chord values for all angles between zero and 180° in half-degree increments.

The next important step in the development of trigonometry occurred in India. Scholars of the 5th century C.E. had by this time discovered that working with half-chords for half-angles greatly simplified the theory of chords and its applications to astronomy. This approach is almost the same construct as the sine function of today. Whereas we think of sine as a ratio of lengths (the length of the half-chord to the radius), Indian scholars interpreted sine as the actual length of the half-chord. Of course, this value of an angle differed for circles of different sizes. The scholars Aryabhata, Bhaskara II, and others developed astonishingly sophisticated techniques for computing half-chord values.

The Arab scholars of the 10th century took a great interest in the work from India. The mathematician Abu al-Wafa (ca. 950) of Baghdad systemized theorems and proofs of Indian trigonometry and prepared his own comprehensive table of half-chord values. He is also believed to have invented the tangent function, which he called the “shadow,” and possibly the secant and the cosecant functions. (Still, all were thought of as specific lengths, not as ratios of lengths.)

In the 12th century, European scholars began translating the Arabic works and soon became acquainted with the extensive theory of trigonometry. The famous scholar Fibonacci (ca 1170–1250) also travelled extensively in the Arab countries and wrote of their trigonometry in his famous work “Practica geometriae”. Around 1464, the German astronomer and mathematician Regiomontanus (1436–76) compiled “De triangulis omnimodis”, a compendium of trigonometry of that time. This work was enormously influential, and over the following decades other texts on the topic appeared. The German mathematician Georg Joachim Rheticus (1514–74) published, in 1551, “Canon doctrinae triangulorum”, which defined, for the first time, all six basic trigonometric functions, and explained how to relate them to right triangles without making any reference to circles. The Danish physician Thomas Fincke (1561–1656) coined the terms “tangent” and “secant” and developed further fundamental trigonometric formulae. The word “trigonometry” itself was invented by the German mathematician Bartholomaeus Pitiscus (1561–1613). By this time, trigonometry was certainly regarded as a worthwhile topic of mathematical pursuit, independent of applications to astronomy.

The subject also proved to be useful in the study of algebra. The French mathematician François Viète (1540–1603) showed, for example, that one could solve certain cubic equations by making trigonometric substitutions. His famous formula for π can be derived with repeated use of trigonometric functions.

Up until this point, sine values, as well as the other trigonometric values, were still regarded as actual line lengths and not ratios of lengths. After the invention of calculus, Leonhard Euler (1707–83) came to realize that it is appropriate to think of sine not as a physical length, but rather as a function of angle independent of length. He suggested that this could be accomplished by scaling all circles under consideration to unit circles, an operation that is equivalent to dividing all quantities by the radius of the circle. Thus, for the first time, all trigonometric quantities came to be thought of as ratios. In 1748 Euler wrote "Introductio in analysin infinitorum", which became the dominating textbook on the topic of trigonometry for the century that followed. It essentially outlines the principles of trigonometry as we regard them today.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 123. Identify the individuals based on the descriptions from the text in Activity 122.

Abu al-Wafa / Aryabhata / Bartholomaeus Pitiscus / Bhaskara II / Claudius Ptolemy / Eratosthenes of Cyrene / Fibonacci / François Viète / Georg Joachim Rheticus / Hipparchus of Rhodes / Leonhard Euler / Regiomontanus / Thales of Miletus / Thomas Fincke

1. A Danish physician who coined the terms tangent and secant and developed further fundamental trigonometric formulae.
2. A famous scholar who travelled extensively in Arab countries and wrote about trigonometry in his work "Practica geometriae."
3. A French mathematician who showed how to solve certain cubic equations by making trigonometric substitutions.
4. A German astronomer and mathematician who compiled "De triangulis omnimodis," a compendium of trigonometry.
5. A German mathematician who invented the term "trigonometry."
6. A German mathematician who published "Canon doctrinae triangulorum," defining all six basic trigonometric functions.
7. A Greek philosopher credited with using similar triangles to determine the height of the Cheops pyramid.
8. A Greek scholar who computed the circumference of the Earth using lengths of shadows and a simple geometric argument on angles.
9. An Indian scholar who developed sophisticated techniques for computing half-chord values, contributing to trigonometry.
10. A mathematician of Baghdad who systemized theorems and proofs of Indian trigonometry, and possibly invented the tangent function.

11. A mathematician who constructed a table of chord lengths for a circle, contributing to the development of trigonometry.
12. A mathematician who wrote an extensive treatise on the theory of chords and their use in obtaining information about spherical triangles.
13. A mathematician who, after the invention of calculus, suggested thinking of sine as a function of angle independent of length.

Activity 124. Determine whether the statements are true or false by quoting from the text in Activity 122.

1. Ancient texts, like the Rhind papyrus and a Babylonian tablet, feature trigonometric problems, including finding pyramid slope angles and tabulating secant values.
2. Claudius Ptolemy's treatise centred on triangles created by straight lines on a sphere's surface.
3. Greek astronomers opted to use measures of straight lengths rather than working directly with angles in their models for celestial motion.
4. Hipparchus of Rhodes played a role in trigonometry by constructing a table of chord lengths for circles.
5. Indian scholars like Aryabhata and Bhaskara II simplified trigonometry in the 5th century by using half-chords, a concept similar to today's cosine function.
6. Thales of Miletus employed congruent triangles to determine the height of the Cheops pyramid by comparing its shadow length to that of a rod in the ground.
7. The Babylonian clay tablet from 1700 B.C.E. includes a table of tangent values for angles between 15° and 30°.
8. The German mathematician Regiomontanus compiled "De triangulis omnimodis," a trigonometry compendium in 1464.
9. The term "trigonometry" is attributed to German mathematician Bartholomaeus Pitiscus (1561–1613).
10. The use of triangles by surveyors, architects, navigators, and astronomers from ancient times contributed to the formation of trigonometry.

Activity 125. In groups, discuss the points. Refer to the text in Activity 122.

1. Explore the historical applications of trigonometry, such as its use by surveyors, architects, navigators, and astronomers. Discuss how these practical applications contributed to the development of trigonometry as a formal subject.
2. Examine the cultural contributions to trigonometry from different civilizations, including ancient Egypt, Babylon, Greece, India, and the Arab world. How did each culture contribute to the evolution of trigonometric concepts and techniques?

3. Trace the evolution of trigonometric functions from their early forms, such as chord lengths, to the more familiar sine and cosine functions. Discuss how different cultures and mathematicians shaped the understanding and representation of trigonometric concepts.
4. Investigate the role of trigonometry in ancient astronomy, as demonstrated by figures like Hipparchus, Ptolemy, and the Indian scholars. How did trigonometric methods contribute to the study of celestial bodies and the development of astronomical models?
5. Explore how mathematical knowledge, particularly trigonometry, was transmitted across cultures. Discuss the impact of translations, travels, and exchanges between different civilizations, such as the interactions between European, Arab, and Indian scholars.
6. Reflect on the dual nature of trigonometry, both as a practical tool for real-world problem-solving and as a theoretical pursuit in mathematics. Discuss examples of how trigonometry was applied in astronomy, surveying, and algebra, and how it contributed to the advancement of mathematical principles.
7. Investigate the conceptual shift from regarding trigonometric values as actual lengths to understanding them as ratios. Discuss the implications of this shift, especially in the context of Euler's insights and the development of calculus.
8. Focus on the contributions of key mathematicians like Fibonacci, Regiomontanus, Rheticus, and Euler to the development of trigonometry. How did their works shape the understanding of trigonometric functions, and what role did they play in advancing mathematical knowledge?
9. Explore the interdisciplinary connections between trigonometry and other branches of mathematics, such as algebra and calculus. Discuss how trigonometric concepts were integrated into solving cubic equations, deriving formulas like π, and contributing to the foundations of calculus.
10. Examine the enduring impact of Euler's "Introductio in analysin infinitorum" as a dominant textbook on trigonometry. Discuss how Euler's work influenced subsequent generations of mathematicians and shaped the principles of trigonometry as they are regarded today.

Activity 126. Choose one point in Activity 125 and elaborate on it in writing. Refer to the text in Activity 122.

"The fact that all mathematics is symbolic logic
is one of the greatest discoveries of our age."

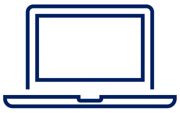
(Bertrand Russell)

Module 4. Areas of Advanced Mathematics

Unit 16. Set Theory

Activity 127. Complete the table with the names of the symbols.

a curly bracket / a round bracket / a square bracket		
Symbol	**British**	**American**
( )	a bracket (1) __________	a parenthesis (pl. parentheses)
[ ]	(2) __________	a bracket
{ }	(3) __________ a brace	



Activity 128. What symbols and signs are used in mathematics? Watch the video "Where Do Math Symbols Come From?" to choose the best answer to the questions. Then watch the video again and make a note of all the mathematical signs and symbols and their purpose. Extend the list with your examples.

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1. Why did Robert Recorde create the equal sign?
A. He wanted to make mathematics more difficult for students
B. He was tired of repeatedly writing "is equal to" in his work
C. He needed a symbol that looked like Greek letters
D. He wanted to replace all words in mathematics with symbols

2. Where does the plus sign for addition come from?
A. A shortened form of the Latin word "et"

B. Two parallel lines that represent equality
C. An exclamation mark used by Christian Kramp
D. A Greek letter that means "more"

3. What is the main reason mathematicians invented symbols?
A. To make mathematics impossible for non-mathematicians to understand
B. To communicate with alien civilizations
C. To avoid repetition and lengthy written explanations
D. To make their work look more impressive and complex

4. How is the capital sigma symbol described?
A. It represents a number that cannot be written in decimal form
B. It shows how many times to repeat a multiplication operation
C. It condenses a long string of sequential terms being added together
D. It indicates when to divide a result by three

5. What can be inferred about mathematical symbols?
A. All symbols were carefully chosen to visually represent their meaning
B. Symbols are a universal language that all civilizations would use identically
C. Understanding mathematical symbols requires memorization and practice, like learning a language
D. Mathematical symbols are more important than understanding the concepts behind them

Activity 129. Match the words with the definitions.

1. finite
2. infinite
3. infinity
4. set
5. subset
6. superset

a. a group of mathematical quantities that have some characteristic in common
b. a set consisting of elements of a given set that can be the same as the given set or smaller
c. a set consisting of elements of a given set that is larger than the given set
d. able to be put in a one-to-one correspondence with part of itself
e. capable of being completely counted
f. the concept of a value greater than any finite numerical value

Activity 130. Read the article. In pairs, discuss the questions in the box.

Loosely speaking, a set is any collection of objects or numbers specified in a well-defined manner. Each item in the set is called an element, or a member, of the set. For example, “dog” is an element of the set of mammals. If an entity “a” is an element of a set S, we write a ∈ S. If “a” does not belong to S, we write a ∉ S.

Sets are typically specified either by listing the elements of the set between a set of braces “{ }”, or listing a few elements of the set to indicate a pattern. For example {a, e, i, o, u} is the set consisting of the five vowels of the alphabet, and {3, 6, 9, 12, ...} is the set of all multiples of 3. It may also be possible to define a set as consisting of elements from some universal collection that satisfy a certain property. For example, {x ∈ R | x > 5} denotes the set of all real numbers that are greater than 5. (Some mathematicians prefer to use a colon “:” instead of a vertical bar in this notation.)

The order in which the elements of a set are listed is immaterial. For example, {A, 6, *} and {*, 6, A} are the same set. Also, elements of a set are listed without repetition. For instance, the set {a, a, a, a, a} is the set with a single element “a”. The empty set (the null set, the void set) is the set that contains no elements.

Two sets are deemed equal if they possess precisely the same elements. For example, the sets {2, 4, 6, 8, ...} and {n | n is a counting number divisible by 2} are equal sets. A set A is said to be a subset of a set B if every element of A is also a member of B. We write A ⊂ B if we are certain that the two sets are not equal, and A ⊇ B if equality of the sets is possible. For example, the set of all multiples of 4 is a subset of the set of all multiples of 2.

Although the intuitive notion of a set as a collection of objects is as ancient as the human race, the idea of a set as a formal mathematical concept was not proposed until the 19th century. In his development of Boolean algebra, The British mathematician George Boole (1815–64) introduced the notion of set as a fundamental tool for the study of formal logic. The German mathematician Georg Cantor (1845–1918), in his attempts to understand the foundation of all of mathematics, came to regard sets as even more basic and fundamental than the notion of number. Cantor properly formalized a theory of set manipulations and introduced the striking notion of cardinality (the cardinality of a set is a measure of the number of elements of the set). His work led him to profound insights into the nature of finite and infinite sets, leading him to extend the concept of number to include more than one type of infinity.

Intuitively, a set is said to be finite if one can recite all the elements of the set in a bounded amount of time. For instance, the set {knife, fork, spoon} is finite, for it takes only a second or two to recite the elements of this set. On the other hand, the set of natural numbers {1, 2, 3, ...} is not finite, for one can never recite each and every element of this set.

Despite our intuitive understanding of the concept, it is difficult to give a precise and direct mathematical definition of a finite set. The easiest approach is to simply define a finite set to be one that is not infinite, since the notion of an infinite set can be made clear. Alternatively, since there is a well-defined procedure for mechanically writing down the string of natural numbers 1, 2, 3, ..., one can define a finite set to be any set S whose elements can be put in one-to-one correspondence with a bounded initial segment of the string of natural numbers. For instance, matching “knife” with 1, “fork” with 2, and “spoon” with 3, the set {knife, fork, spoon} is finite because its elements can be matched precisely with the string of natural numbers {1, 2, 3}.

In 1902 the British mathematician and philosopher Bertrand Arthur William Russell (1872–1970) stunned the mathematical community with his construction of a simple paradox, today called Russell’s paradox, that shows that our naive understanding of the notion of set is fundamentally flawed. Although Cantor believed that set theory is the foundation on which all of mathematics is built, it became clear to mathematicians that the concept of a set and what it means to be an “element of” must remain as undefined terms. In the decades that followed, mathematicians such as Ernst Friedrich Ferdinand Zermelo (1871–1953) attempted to develop an axiomatic theory of sets (based on undefined terms) that successfully avoids Russell’s paradox. To this day, not all mathematicians agree that this goal has yet been achieved.

(from Encyclopaedia Britannica)

1. What is a set?
2. How are sets typically specified?
3. What is the significance of the order in which elements are listed in a set?
4. What types of sets are there?
5. Who introduced the notion of sets?
6. How did George Boole use sets in the development of Boolean algebra?
7. What did Georg Cantor contribute to the understanding of sets and the concept of infinity?
8. How is the finiteness of a set intuitively described, and why is it challenging to provide a precise mathematical definition?

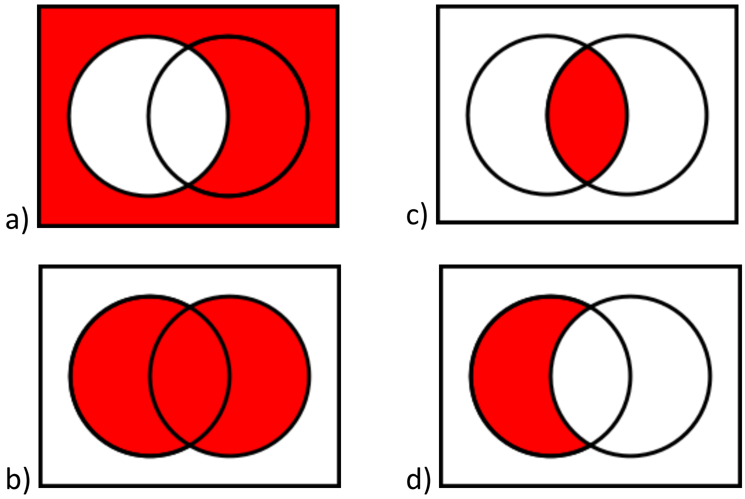
Activity 131. Reorder the sentences to make a text on set theory.

a. A finite set has a definite number of members; such a set might consist of all the integers from 1 to 1,000 or all marked bus stops along a given route.
b. A set is commonly represented by a list of its members, or elements, enclosed within braces; the statement that a set called A comprises the numbers 1, 2, and 3 is made by the expression A = {1, 2, 3}.
c. A set that has no members is called an empty set (or a null or void set) and is denoted by the symbol Ø.

d. An infinite set has an endless number of members; all the positive integers or all points along a given line compose infinite sets.
e. Set theory is a branch of mathematics that deals with the properties of well-defined collections of objects, which may be of a mathematical nature, such as numbers or functions, or not.
f. Sets may be finite or infinite.

Activity 132. Match the names of set operations with the Venn diagrams.

1) set complement
2) set difference
3) set intersection
4) set union



Activity 133. In writing, comment on the quote from the text in Activity 130, justifying the author's position as well as expressing your stance.

"In his attempts to understand the foundation of all of mathematics, the German mathematician Georg Cantor came to regard sets as more basic and fundamental than the concept of number."

Unit 17. Mathematical Logic

Activity 134. Complete the table with the types of mathematical statements.

axiom / conjecture / corollary / lemma / postulate / theorem	
Statement	**Example**
(1) __________ (2) __________ (Assumption)	0 is a natural number.
(3) __________ (Hypothesis)	Every even natural number greater than 2 is the sum of two prime numbers.
(4) __________ (Proposition)	The sum of the squares on the legs (catheti) of a right triangle is equal to the square on the hypotenuse (the side opposite the right angle).
(5) __________	If a prime "p" divides the product "ab" of two integers "a" and "b", then "p" must divide at least one of those integers "a" or "b".
(6) __________	All internal angles in a rectangle are right angles. All internal angles in a square are right angles.

Activity 135. Read the two paragraphs. Conclude which one explores deductive reasoning and which — inductive.

1. In the scientific method, there are two general processes for establishing results. The first, called __________ reasoning, arrives at general conclusions by observing specific examples, identifying trends, and generalizing. "The sun has always risen in the past, therefore it will rise tomorrow," for example, illustrates this mode of reasoning. The __________ process relies on discerning patterns but does not attempt to prove that the patterns observed apply to all cases. (Maybe the sun will not rise tomorrow.) For this reason, a conclusion drawn by the __________ process is called a conjecture or an educated guess. If there is just one case for which the conclusion does not hold, then the conjecture is false. Such a case is called a counterexample.

2. On the other hand, __________ reasoning works to prove a specific conclusion from one or more general statements using logical reasoning (as given by formal logic) and valid

arguments. For example, given the statements, “All cows eat grass” and “Daisy is a cow,” we can conclude, by __________ reasoning, that Daisy eats grass. __________ reasoning does not rely on the premises that are made necessarily being true. For example, “Sydney and Boston are planets, therefore Boston is a planet” is a valid argument, whereas “Either Boston or Venus is a planet, therefore Venus is a planet” is invalid.

Activity 136. Complete the sentences with the words from the text in Activity 137.

1. In mathematics, the systematic study of __________ is called formal logic or symbolic logic.
2. It analyzes the structure of arguments, as well as the methods and validity of mathematical deduction and __________.
3. He sought to identify modes of __________ that are valid by virtue of their structure, not their content.
4. This mode of thought allowed Euclid (ca. 300–260 B.C.E.) to formalize geometry, using deductive __________ to __________ geometric truths from a small collection of axioms (self-evident truths).
5. Propositional logic is the part which deals with __________ involving simple declarative sentences (statements) joined by connectives.
6. Beginning with an impressively minimal set of __________ (“self-evident” logical principles), they attempted to establish the logical foundations of all of mathematics.

Activity 137. Read the article. Determine whether the statements in the box below are true or false by quoting from the text.

In mathematics, the systematic study of reasoning is called formal logic or symbolic logic. It analyzes the structure of arguments, as well as the methods and validity of mathematical deduction and proof.

The principles of logic can be attributed to Aristotle (384–322 B.C.E.), who wrote the first systematic treatise on the subject. He sought to identify modes of inference that are valid by virtue of their structure, not their content. For example, “Green and blue are colours; therefore green is a colour” and “Cows and pigs are reptiles; therefore cows are reptiles” have the same structure (“A and B, therefore A”), and any argument made via this structure is logically valid. (In particular, the second example is logically sound.) This mode of thought allowed Euclid (ca. 300–260 B.C.E.) to formalize geometry, using deductive proofs to infer geometric truths from a small collection of axioms (self-evident truths).

No significant advance was made in the study of logic for the millennium that followed. This period was mostly a time of consolidation and transmission of the material from antiquity. The Renaissance, however, brought renewed interest in the topic. Mathematical scholars of the time, Pierre Hérigone and Johann Rahn in particular, developed means for representing logical arguments with abbreviations and symbols, rather than words and sentences. Gottfried Wilhelm Leibniz (1646–1716) came to regard logic as “universal mathematics.” He advocated the development of a “universal language” or a “universal calculus” to quantify the entire process of mathematical reasoning. He hoped to devise new mechanical symbolism that would reduce errors in thinking to the equivalent of arithmetical errors. (He later abandoned work on this project, assessing it too daunting a task for a single man.)

In the mid-1800s George Boole succeeded in creating a purely symbolic approach to propositional logic, that part which deals with inferences involving simple declarative sentences (statements) joined by the connectives: not, and, or, if ... then ..., iff (if and only if). (These are called the negation, conjunction, disjunction, conditional, and the biconditional, respectively.) He successfully applied it to mathematics, thereby making a significant step to achieving Leibniz’s goal.

In 1879 the German mathematician and philosopher Gottlob Frege constructed a symbolic system for predicate logic. This generalizes propositional logic by including quantifiers: statements using words such as “some”, “all”, and “no”. (For example, “All men are mortal” as opposed to “This man is mortal.”) At the turn of the century David Hilbert sought to devise a complete, consistent formulation of all of mathematics using a small collection of symbols with well-defined meanings. The English mathematician and philosopher Bertrand Russell, in collaboration with his colleague Alfred North Whitehead, took up Hilbert’s challenge. In 1925 they published a monumental work. Beginning with an impressively minimal set of premises (“self-evident” logical principles), they attempted to establish the logical foundations of all of mathematics. This was an impressive accomplishment. (After hundreds of pages of symbolic manipulations, they established the validity of “1 + 1 = 2,” for example.) Although they did not completely reach their goal, Russell and Whitehead’s work has been important for the development of logic and mathematics.

Six years after the publication of their efforts, however, Kurt Gödel stunned the mathematical community by proving Hilbert’s (and Leibniz’s) goal to be futile. He demonstrated once and for all that any formal system of logic sufficiently sophisticated to incorporate basic principles of arithmetic cannot attain all the statements it hopes to prove. His results are today called Gödel’s incompleteness theorems. The vision to reduce all truths of reason to incontestable arithmetic was thereby shattered.

(from Encyclopaedia Britannica)

1. Aristotle wrote the first treatise on logic, focusing on the content rather than the structure of arguments.
2. Bertrand Russell and Alfred North Whitehead's work in 1925 attempted to establish the logical foundations of mathematics.
3. David Hilbert aimed to formulate all of mathematics consistently, inspiring Russell and Whitehead's monumental 1925 work.
4. Formal logic, also known as symbolic logic, systematically studies reasoning in mathematics, analyzing argument structures and the validity of deductions and proofs.
5. George Boole's symbolic approach to propositional logic had no significant impact on mathematics.
6. In 1879, Gottlob Frege developed a symbolic system for predicate logic, extending propositional logic with quantifiers like "some" and "all."
7. Kurt Gödel's incompleteness theorems supported Hilbert's goal.
8. Leibniz regarded logic as "universal mathematics" and advocated for the development of a "universal language" in mathematical reasoning.
9. The Renaissance was a period of decline for logical studies.
10. The vision to reduce all truths of reason to incontestable arithmetic remains intact despite Gödel's results.

Activity 138. Reorder the sentences to make a text on foundations of mathematics.

a. He too searched for small collections of concepts that were fundamental and, hopefully, common to all fields.
b. In the late 1800s and at the turn of the century with the discovery of Russell's paradox in set theory, mathematicians were led to apparent paradoxes and inconsistencies within the seemingly very basic notions of "set" and "number."
c. Leonhard Euler (1707–83) produced fundamental results in disparate branches of mathematics and often saw connections between those branches.
d. The branch of mathematics concerned with the justification of mathematical rules, axioms, and modes of inference is called foundations of mathematics.
e. The paradigm for critical mathematical analysis came from the work of the great geometer Euclid (ca. 300–260 B.C.E.) who, in his work "The Elements", demonstrated that all geometry known at his time can be logically deduced from a small set of self-evident truths (axioms).
f. This led to the fervent study of the fundamental principles of elementary mathematics and even to the study of the process of mathematical thinking itself (formal logic).

g. Understanding the philosophical foundations of mathematics is still an area of intense scholarly research.

Activity 139. Are there unanswerable questions and unsolvable problems to mathematics? If so, give examples. Watch the video "The Paradox at the Heart of Mathematics: Gödel's Incompleteness Theorem" to choose the best answer to the questions. Describe how Gödel's discovery upended mathematics.

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1. What paradox inspired Kurt Gödel's mathematical discovery?
A. A statement that refers to itself and creates a logical contradiction
B. A complex equation that nobody could solve
C. An ancient Greek mathematical problem
D. A contradiction in basic arithmetic operations

2. How did Gödel make mathematics "talk about itself"?
A. He wrote mathematical statements in different languages
B. He created new axioms that described other axioms
C. He translated mathematical statements into code numbers
D. He used words instead of numbers in equations

3. According to Gödel's incompleteness theorem, what is true about axiomatic systems?
A. They can all be completed if enough axioms are added
B. They always contain true statements that cannot be proved
C. They only work for simple mathematical problems
D. They were proven wrong by modern computers

4. How did most mathematicians respond to Gödel's discovery?
A. They accepted the new reality it presented
B. They proved that his theorem was incorrect
C. They stopped working on mathematical proofs
D. They ignored it completely and continued as before

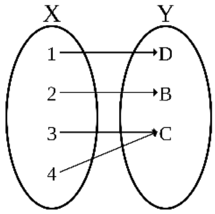
5. What positive outcome resulted from Gödel's theorem?
A. It allowed mathematicians to prove every mathematical claim
B. It eliminated all paradoxes from mathematics
C. It inspired important developments in early computers
D. It created more than 350 proofs of the Pythagorean theorem

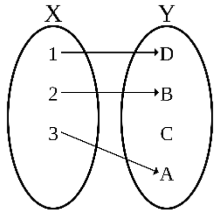
Activity 140. Single out the keywords (key phrases) from the text in Activity 137 and write an abstract of it.

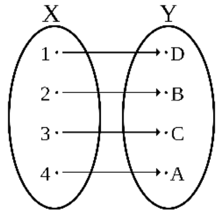
Unit 18. Function Theory

Activity 141. Match the functions with the diagrams.

1) bijection (bijective, one-to-one)
2) injection (injective, one-to-one)
3) surjection (surjective, onto)

a) 

b) 

c) 

Activity 142. Match the words with the definitions.

1. codomain
2. dependent variable
3. domain
4. graph
5. independent variable
6. range
7. value

a. a drawing or a visual representation that shows the relationship between two or more variables
b. a particular magnitude, number, or quantity
c. a variable in a functional relation whose value determines the value or values of other variables
d. a variable in a functional relation whose value is determined by the values assumed by other variables in the relation
e. the set of all possible outputs for a given function
f. the set of values of the independent variable of a function for which the functional value exists
g. the set of values that a function is allowed to take

Activity 143. Find the definitions of the term "function" in the paragraphs of Activity 144 proposed by the mathematicians.

1. Georg Cantor
2. Gottfried Wilhelm Leibniz
3. Jean Baptiste Joseph Fourier
4. Leonhard Euler
5. Nicole Oresme
6. Peter Gustav Lejeune Dirichlet

Activity 144. Reorder the paragraphs to make a text on function theory.

a. Advanced texts in mathematics today typically present all three definitions of a function — as a formula, as a set of ordered pairs, and as a mapping — and mathematicians will typically work with all three approaches.
b. In 1694 the German mathematician Gottfried Wilhelm Leibniz, codiscoverer of calculus, coined the term "function" to mean the slope of the curve, a definition that has very little in common with our current use of the word. The great Swiss mathematician Leonhard Euler (1707–83) recognized the need to make the notion of a relationship between quantities explicit, and he defined the term "function" to mean a variable quantity that is dependent upon another quantity. Euler introduced the notation *f*(x) for "a function of x," and promoted the idea of a function as a formula. He based all his work in calculus and analysis on this idea, which paved the way for mathematicians to view trigonometric quantities and logarithms as functions. This notion of function subsequently unified many branches of mathematics and physics.
c. In 1822 the French physicist and mathematician Jean Baptiste Joseph Fourier presented work on heat flow. He represented functions as sums of sine and cosine functions but commented that such representations may be valid only for a certain range of values. This later led the German mathematician Peter Gustav Lejeune Dirichlet (1805–1859) to propose a more precise definition: A function is a correspondence that assigns a unique value of a dependent variable to every permitted value of an independent variable. This, on an elementary level, is the definition generally accepted today.
d. In the late 19^{th} century, the German mathematician Georg Cantor (1845–1918) attempted to base all of mathematics on the fundamental concept of a set. Because the terms variable and relationship are difficult to specify, Cantor proposed an alternative definition of a function: A function is a set of ordered pairs in which every first element is different.
e. In the mid-1300s the French mathematician Nicole Oresme discovered that a uniformly varying quantity (such as the position of an object moving with uniform

velocity, for instance) could be represented pictorially as a graph, and that the area under the graph represents the total change of the quantity. Oresme was the first to describe a way of graphing the relationship between an independent variable and a dependent one and, moreover, demonstrate the usefulness of the task.

f. Mathematicians consequently came to think of functions as “mappings” that assign to elements of one set X, called the domain of the function, elements of another set Y, called the codomain. (Each element “x” of X is assigned just one element of Y.) One can thus depict a function as a diagram of arrows in which an arrow is drawn from each member of the domain to its corresponding member of the codomain. The function is then the complete collection of all these correspondences.

g. Since the time of antiquity, scholars were interested in identifying rules or relationships between quantities. For example, the ancient Egyptians were aware that the circumference of a circle is related to its diameter via a fixed ratio that we now call “pi”, and Chinese scholars, and later the Pythagoreans, knew that the three sides of a right triangle satisfy the simple relationship given by Pythagoras’s theorem. Although these results were not expressed in terms of formulae and symbols (the evolution of algebraic symbolism took many centuries), scholars were aware that the value of one quantity could depend on the value of other quantities under consideration. Although not explicit, the notion of a “function” was in mind.

h. This idea is based on the fact that the graph of a function is nothing more than a collection of points (x,y) with no two y-values assigned to the same x-value. Cantor’s definition is very general and can be applied not only to numbers but to sets of other things as well.

Activity 145. Single out the keywords (key phrases) from the text in Activity 144 and write an abstract of it.

Activity 146. In groups, discuss the points. Refer to the text in Activity 144.

1. Discuss how the understanding of mathematical concepts, such as the notion of a "function," has evolved over time, from ancient civilizations to the modern era. What role did different cultures and mathematicians play in shaping these concepts?
2. Explore the significance of Nicole Oresme's discovery in the mid-1300s regarding the representation of a uniformly varying quantity as a graph. How did this visual representation contribute to the understanding of relationships between independent and dependent variables?
3. Analyze the diverse definitions of a "function" proposed by Leibniz, Euler, Dirichlet, and Cantor. How did these definitions reflect the mathematical thinking and

challenges of their respective time periods? In what ways did these definitions contribute to the unification of mathematical branches?

4. Examine the role of notation, specifically Euler's introduction of *f*(x) for "a function of x," in formalizing mathematical concepts. How did the use of symbols and formulae contribute to the development of calculus and analysis?
5. Delve into Georg Cantor's attempt to base all of mathematics on the concept of a set and his alternative definition of a function as a set of ordered pairs. How did this set-based perspective impact the understanding of functions?
6. Discuss the modern conceptualization of functions as "mappings" between sets, with a domain and a codomain. How does this perspective offer a unified understanding of functions, and what advantages does it provide in contemporary mathematics?
7. Consider the approach of presenting all three definitions of a function — as a formula, as a set of ordered pairs, and as a mapping — in advanced mathematics texts. How does this integrated approach enhance the comprehension and application of mathematical concepts?
8. Explore the practical implications of the historical developments in mathematical understanding, particularly in terms of how mathematicians work with and apply the concept of a function in various fields and disciplines.

Activity 147. Choose one point in Activity 146 and elaborate on it in writing. Refer to the text in Activity 144.

Unit 19. Mathematical Analysis and Calculus

Activity 148. Read the passage. In pairs, discuss the questions in the box.

Any topic in mathematics that makes use of the notion of a limit in its study is called analysis. Calculus comes under this heading, as does the summation of infinite series, and the study of real numbers. The Greek mathematician Pappus of Alexandria (ca. 320 C.E.) called the process of discovering a proof or a solution to a problem "analysis." He wrote about "a method of analysis" somewhat vaguely in his geometry text "Collection", which left mathematicians centuries later wondering whether there was a secret method hidden behind all of Greek geometry. The great René Descartes (1596–1650) developed a powerful method of using algebra to solve geometric problems. His approach became known as analytic geometry. The branch of mathematics that deals with the notion of continuous growth and change is called calculus (infinitesimal calculus). It is based on the concept of infinitesimals, exceedingly small quantities, and on the concept of a limit, quantities that can be approached more and more closely but never reached. The branch of calculus known as differential calculus deals with notions of slope, rates of change and ratios of change. For example, a study of velocity, which can be described as the rate of change of position, falls under the study of differential calculus, as do other concepts that arise in the study of motion. Any process that involves segmenting a quantity into manageable pieces, summing, and taking the limit of these sums as the process is refined falls under the category of integral calculus. The word "calculus" comes from the Latin word "calx" for "pebble," which in turn is derived from the Greek word "chalis" for "limestone." Small beads or stones arranged in a counting board or on an abacus were often used to aid mathematical calculations, and the word "calculus" came to refer to all mathematical activity. Today, however, the word is used almost exclusively to denote the study of continuous change.

(from Encyclopaedia Britannica)

1. Define the terms "limit" and "infinitesimal".
2. What is the origin of the word "calculus"?
3. What is the difference between analysis and calculus?
4. How has the use of the words "analysis" and "calculus" evolved?
5. Distinguish between infinitesimal calculus, differential calculus, and integral calculus.

Activity 149. Do the quiz on calculus. In pairs, compare your answers.

1. **When did the study of calculus begin?**
    a. ancient times
    b. the 17^{th} century
    c. the 18^{th} century
    d. the 19^{th} century
2. **Who among the ancient Greek scholars is mentioned for their work on infinitesimals?**
    a. Plato
    b. Pythagoras
    c. Archimedes
    d. Eudoxus
3. **What method was developed by Greek scholars to compute the area or volume of a curved figure?**
    a. method of indivisibles
    b. method of limits
    c. method of exhaustion
    d. method of infinitesimals
4. **Which mathematician wrote the first textbook on integration methods in 1635?**
    a. Johannes Kepler
    b. Bonaventura Cavalieri
    c. Pierre de Fermat
    d. John Wallis
5. **In the mid-1600s, what breakthrough united the study of tangent problems and area problems?**
    a. inverse relationship discovery
    b. fundamental theorem of calculus
    c. method of exhaustion
    d. Kepler's optimization solutions
6. **How did Newton refer to the quantity being studied in calculus and its rate of change?**
    a. fluent and fluxion
    b. element and derivative
    c. variable and gradient
    d. integral and differential
7. **Who developed a notational system for calculus accessible to a wide audience in the 17^{th} century?**

a. Isaac Newton
b. Pierre de Fermat
c. Blaise Pascal
d. Gottfried Wilhelm Leibniz

8. **Which 18th-century mathematician questioned the validity of calculus in his scathing essay "The Analyst"?**
   a. Leonhard Euler
   b. George Berkeley
   c. Karl Weierstrass
   d. Augustine Louis Cauchy
9. **Who is credited with developing a concept of integration applied to a wider class of functions in the 19th century?**
   a. Bernhard Riemann
   b. Joseph-Louis Lagrange
   c. Henri Léon Lebesgue
   d. Pierre-Simon Laplace
10. **What idea did Augustine Louis Cauchy propose to replace the notion of an infinitesimal?**
    a. method of limits
    b. method of exhaustion
    c. measure theory
    d. fluxion theory

Activity 150. Read the article. Review your answers to the quiz in Activity 149.

The study of calculus begins with the study of motion, a topic that has fascinated and befuddled scholars since the time of antiquity. The first recorded work of note in this direction dates back to the Greek scholars Pythagoras (ca. 569–475 B.C.E) and Zeno of Elea (ca. 500 B.C.E.), and their followers, who put forward the notion of an infinitesimal as one possible means for explaining the nature of physical change. Motion could thus possibly be understood as the aggregate effect of a collection of infinitely small changes. Zeno, however, was very much aware of fundamental difficulties with this approach and its assumption that space and time are consequently each continuous and thus infinitely divisible. Through a series of ingenious logical arguments, Zeno reasoned that this cannot be the case. At the same time, Zeno presented convincing reasoning to show that the reverse position, that space is composed of fundamental indivisible units, also cannot hold. The contradictory issues proposed by Zeno were not properly resolved for well over two millennia.

The concept of the infinitesimal also arose in the ancient Greek study of area and volume. Scholars of the schools of Plato (428–348 B.C.E.) and of Eudoxus of Cnidus (ca. 370 B.C.E.) developed a "method of exhaustion," which attempted to compute the area or volume of a curved figure by confining it between two known quantities, both of which can be made to resemble the desired object with any prescribed degree of accuracy. Archimedes of Syracuse (287–212 B.C.E.) applied this method to compute the area of a section of a parabola, and 600 years later, Pappus of Alexandria (ca. 300–350 C.E.) computed the volume of a solid of revolution via this technique. Although successful in computing the areas and volumes of a select collection of geometric objects, scholars had no general techniques that allowed for the development of a general theory of area and volume. Each individual calculation for a single specific example was hailed as a great achievement in its own right.

The resurgence of scientific investigation in the mid-1600s led European scholars to push the method of exhaustion beyond the point where Archimedes and Pappus had left it. Johannes Kepler (1571–1630) extended the use of infinitesimals to solve optimization problems. Others worked on the problem of finding tangents to curves, an important practical problem, and the problem of finding areas of irregular figures. In 1635, the Italian mathematician Bonaventura Cavalieri wrote the first textbook on what we would call "integration methods". He described a general "method of indivisibles" useful for computing volumes. The principle today is called Cavalieri's principle.

The French mathematician Gilles Personne de Roberval (1602–75) was the first to link the study of motion to geometry. He realized that the tangent line to a geometric curve could be interpreted as the instantaneous direction of motion of a point travelling along that curve. The philosopher and mathematician René Descartes (1596–1650) developed general techniques for finding the formula for the tangent line to a curve at a given point. This technique was later picked up by Pierre de Fermat (1601–65), who used the study of tangents to solve maxima and minima problems in much the same way we solve such problems today. As a separate area of study, Fermat also developed techniques of integral calculus to find areas between curves and lengths of arcs of curves, which were later developed further by Blaise Pascal (1623–62) and English mathematicians John Wallis (1616–1703) and Isaac Barrow (1630–77).

At the same time scholars, including Wallis, began studying series and infinite products. The Scottish mathematician James Gregory (1638–75) developed techniques for expressing trigonometric functions as infinite sums, thereby discovering Taylor series 40 years before Brook Taylor (1685–1731) independently developed the same results.

By the mid-1600s, certainly, all the pieces of calculus were in place. Yet scholars at the time did not realize that all the varied problems being studied belonged to one unified whole, namely, that the techniques used to solve tangent problems could be used to solve area problems, and vice versa. A fundamental breakthrough came in the 1670s when, independently, Gottfried Wilhelm Leibniz (1646–1716) of Germany and Sir Isaac Newton

(1642–1727) of England discovered an inverse relationship between the "tangent problem" and the "area problem." The discovery of the fundamental theorem of calculus brought together the disparate topics being studied, provided a beautiful and natural perspective on the subject as a whole, and allowed scholars to make significant advances in solving geometric and physical problems with spectacular success. Despite the content of knowledge that had been established up until that time, it is the discovery of the fundamental theorem of calculus that represents the discovery of calculus.

Newton approached calculus through a concept of "flowing entities." He called any quantity being studied a "fluent" and its rate of change a "fluxion". Records show that he had developed these ideas as early as 1665, but he did not publish an account of his theory until 1704. Unfortunately, his writing style and choice of notation also made his version of calculus accessible only to a select audience. Leibniz, on the other hand, made explicit use of an infinitesimal in his development of the theory. He called the infinitesimal change of a quantity "x" a differential, denoted "dx". Leibniz invented a beautiful notational system for the subject that made reading and working with his account of the theory immediately accessible to a wide audience. (Many of the symbols we use today in differential and integral calculus are due to Leibniz.) Leibniz formulated his approach in the mid-1670s and published his account of the subject in 1684. Although it is now known that Newton and Leibniz had made their discoveries independently, matters at the time were not clear, and a bitter dispute arose over the priority for the discovery of calculus.

Applying the techniques to problems of the real world became the main theme of 18th-century mathematics. Newton's famous 1687 text "Principia" paved the way with its analysis of the laws of motion and the mechanics of the solar system. The Swiss brothers Jakob Bernoulli (1654–1705) and Johann Bernoulli (1667–1748) of the famous Bernoulli family, champions of Leibniz in the famous dispute, studied the newly invented calculus and were the first to give public lectures on the topic. Johann Bernoulli was hired to teach differential calculus to the French nobleman Guillaume François de L'Hopital (1661–1704) via written correspondence. In 1696 L'Hopital then published the content of Johann's letters with his own name as author. The Italian mathematician Marie Gaetana Agnesi (1718–99) wrote the first comprehensive textbook dealing with both differential and integral calculus in 1755.

The Swiss mathematician Leonhard Euler (1707–83) and French mathematicians Joseph-Louis Lagrange (1736–1813) and Pierre-Simon Laplace (1749–1827) were prominent in developing the theory of differential equations. Euler also wrote extensively on the subject of calculus, showing how the theory can be applied to a vast range of pure and applied mathematical problems. Yet despite the evident success of calculus, some 18th-century scholars questioned the validity and the soundness of the subject.

The sharpest critic of Newton's and Leibniz's work was the Anglican Bishop of Coyne, George Berkeley (1685–1753). In his scathing essay, "The Analyst," Berkeley demonstrated, convincingly, that both Newton's notion of a fluxion and Leibniz's concept of an infinitesimal are ill-defined, and that the foundations of the subject are consequently insecure. Mathematicians consequently began looking for ways to put calculus on a sound footing. Significant progress was not made until the 19^{th} century, when the French mathematician Augustine Louis Cauchy (1789–1857) suggested that the notion of an infinitesimal should be replaced by that of a limit. German mathematician Karl Weierstrass (1815–97) developed this idea further and was the first to give absolutely clear and precise definitions to all concepts used in calculus, devoid of any mystery or reliance on geometric intuition. The work of the German mathematician Richard Dedekind (1831–1916) highlighted the role properties of the real number system play in ensuring the validity of the intermediate-value theorem and extreme-value theorem and all the essential results that follow from them.

Initially, calculus was deemed a theory pertaining only to continuous change and continuous functions. The German mathematician Bernhard Riemann (1826–66) was the first to consider, and give careful discussion on, the integration of discontinuous functions. His definition of an integral is the one typically presented in textbooks today. At the end of the 19^{th} century, the French mathematician Henri Léon Lebesgue (1875–1941) literally turned Riemann’s approach around and developed a concept of integration that can be applied to a much wider class of functions and class of settings. In order to do this, Lebesgue had to develop a general “measure theory” for determining the size of complicated sets. His new theory proved to be fundamentally important, and it now has profound applications to a wide range of mathematical topics. It proved to be especially important to the sound development of probability theory.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 151. Identify the individuals based on the descriptions from the text in Activity 150.

Archimedes of Syracuse / Augustine Louis Cauchy / Bernhard Riemann / Blaise Pascal / Bonaventura Cavalieri / Eudoxus of Cnidus / George Berkeley / Gilles Personne de Roberval / Gottfried Wilhelm Leibniz / Guillaume François de L’Hopital / Henri Léon Lebesgue / Isaac Barrow / Jakob Bernoulli / James Gregory / Johann Bernoulli / Johannes Kepler / John Wallis / Joseph-Louis Lagrange / Karl Weierstrass / Leonhard Euler / Marie Gaetana Agnesi / Pappus of Alexandria / Pierre de Fermat / Pierre-Simon Laplace / Plato / Pythagoras / René Descartes / Richard Dedekind / Sir Isaac Newton / Zeno of Elea

1. The Anglican Bishop of Coyne, a critic of Newton's and Leibniz's work on calculus.

2. English mathematician who contributed to the study of tangents and integral calculus.
3. English mathematician who, independently of Leibniz, discovered the fundamental theorem of calculus.
4. Extended the use of infinitesimals to solve optimization problems in the mid-1600s.
5. French mathematician known for his work on celestial mechanics and contributions to calculus.
6. French mathematician who contributed to the theory of differential equations.
7. French mathematician who developed a concept of integration applied to a wider class of functions, introducing measure theory.
8. French mathematician who linked the study of motion to geometry.
9. French mathematician who suggested replacing infinitesimals with limits in calculus.
10. French nobleman who learned calculus from Johann Bernoulli and published the content of Johann's letters.
11. German mathematician who considered the integration of discontinuous functions.
12. German mathematician who provided clear and precise definitions for calculus concepts.
13. German mathematician who, independently of Newton, discovered an inverse relationship between the "tangent problem" and the "area problem."
14. German mathematician whose work highlighted the role of the real number system in calculus.
15. Greek mathematician who applied the "method of exhaustion" to compute the area of a section of a parabola.
16. Greek philosopher associated with the school of thought studying area and volume.
17. Greek scholar known for his ingenious logical arguments regarding the difficulties of the infinitesimal approach and the nature of space and time.
18. Greek scholar linked to the development of the "method of exhaustion" for computing areas and volumes.
19. Greek scholar who, along with Zeno of Elea, contributed to the notion of an infinitesimal in explaining physical change.
20. Italian mathematician who wrote the first comprehensive textbook dealing with both differential and integral calculus.
21. Italian mathematician who wrote the first textbook on "integration methods" and introduced the "method of indivisibles."
22. Mathematician who computed the volume of a solid of revolution using the "method of exhaustion."
23. Mathematician who further developed integral calculus techniques.
24. Mathematician who used the study of tangents to solve maxima and minima problems and developed integral calculus techniques.
25. Philosopher and mathematician who developed general techniques for finding the formula for the tangent line to a curve.

26. Scottish mathematician who developed techniques for expressing trigonometric functions as infinite sums.
27. Swiss mathematician and brother of Jakob Bernoulli, who studied calculus and taught differential calculus to L'Hopital.
28. Swiss mathematician prominent in developing the theory of differential equations and extensively wrote on calculus.
29. Swiss mathematician who, along with Johann Bernoulli, studied and lectured on calculus.

Activity 152. Rearrange the events in chronological order according to the text of Activity 150. Provide dates where possible.

a. Bishop Berkeley questions the foundations of calculus.
b. Bonaventura Cavalieri introduces "method of indivisibles."
c. Cauchy suggests replacing infinitesimals with limits.
d. Cavalieri writes the first textbook on integration methods.
e. Dedekind emphasizes the role of the real number system in calculus.
f. Descartes, Fermat, Wallis, and Barrow contribute to tangent problems and areas of curves.
g. Greek scholars Pythagoras and Zeno of Elea propose the notion of an infinitesimal to explain physical change.
h. Greek scholars, including Plato, Eudoxus, and Archimedes, develop the "method of exhaustion" for computing areas and volumes.
i. Johannes Kepler extends infinitesimals to solve optimization problems.
j. Lebesgue develops a new concept of integration, introducing measure theory.
k. Lebesgue's theory profoundly impacts probability theory.
l. Leibniz and Newton independently discover the fundamental theorem of calculus, unifying tangent and area problems.
m. Newton's "Principia" analyzes laws of motion and solar system mechanics.
n. Riemann considers integration of discontinuous functions.
o. Roberval links the study of motion to geometry.
p. The Bernoulli brothers, L'Hopital, and Agnesi contribute to the application and teaching of calculus.
q. Weierstrass provides clear and precise definitions for calculus concepts.

Activity 153. Determine whether the statements are true or false by quoting from the text in Activity 150.

1. Calculus starts by examining the concept of infinity.
2. In the 1650s, Leibniz and Newton independently found a crucial connection between the "tangent problem" and the "area problem" in calculus.
3. Infinitesimals were employed by Johannes Kepler for solving optimization problems.
4. Initially, calculus was considered a theory unrelated to continuous change and continuous functions.
5. Leonhard Euler played a significant role in advancing the theory of differential equations.
6. Newton's approach to calculus involved the idea of "flowing entities."
7. Scholars possessed comprehensive techniques enabling the development of a general theory of area and volume.
8. The Anglican Bishop of Coyne, George Berkeley, supported Newton's and Leibniz's work.
9. The resurgence of scientific investigation influenced European scholars to extend the method of exhaustion beyond the point left by Archimedes and Pappus.
10. The Swiss brothers Jakob Bernoulli and Johann Bernoulli opposed Leibniz in the famous dispute.

Activity 154. In groups, discuss the points. Refer to the text in Activity 150.

1. How did the study of calculus evolve from the ancient Greeks' exploration of infinitesimals to the development of integral calculus and the fundamental theorem of calculus in the 17^{th} century?
2. Explore the "method of exhaustion" employed by Greek scholars and its application in computing areas and volumes. How did Archimedes and Pappus contribute to this method?
3. Discuss Cavalieri's principle as an early method of computing volumes and its significance in the development of calculus. How did scholars extend and refine these methods in the 17^{th} century?
4. Analyze the contributions of Gilles Personne de Roberval, René Descartes, and Pierre de Fermat in linking the study of motion to geometry. How did these developments lay the groundwork for calculus?
5. Investigate the 17^{th}-century scholars' exploration of series and infinite products, particularly the contributions of James Gregory. How did this contribute to the overall understanding of calculus?
6. Discuss the realization in the 17^{th} century that the techniques used to solve tangent problems could also be applied to solve area problems, leading to the fundamental theorem of calculus. How did this unify the disparate topics in calculus?

7. Explore the Newton-Leibniz dispute over the priority of the discovery of calculus. How did Newton's concept of "fluxions" compare to Leibniz's use of infinitesimals, and how did this controversy impact the development of calculus?
8. Investigate how 18th-century mathematicians, including Jakob and Johann Bernoulli, applied calculus to real-world problems, as seen in Newton's "Principia" and the public lectures on calculus.
9. Examine the challenges and criticisms posed by scholars like George Berkeley in the 18th century. How did these critiques prompt a search for a more rigorous foundation for calculus?
10. Explore the 19th-century developments by mathematicians like Cauchy, Weierstrass, and Dedekind in putting calculus on a sound footing. How did the introduction of limit concepts and precise definitions contribute to the development of calculus?

Activity 155. Choose one point in Activity 154 and elaborate on it in writing. Refer to the text in Activity 150.

Unit 20. Probability Theory and Mathematical Statistics

Activity 156. Match the words with the definitions.

1. census
2. event
3. expected value
4. inference
5. life table
6. outcome
7. probability
8. sample space

a. a complete enumeration of a population, typically including details about individuals' characteristics
b. a measure of the likelihood of an event occurring, expressed as a number between 0 and 1
c. a possible result of an experiment or random process
d. a subset of the sample space, representing a particular outcome or set of outcomes
e. a table showing the probability of survival and mortality at different ages in a population
f. the average value of a random variable, calculated as the sum of all possible values multiplied by their respective probabilities
g. the process of drawing conclusions about a population based on a sample of data from that population
h. the set of all possible outcomes of an experiment

Activity 157. What are the Wright brothers, Orville and Wilbur, credited with? Watch the video "The Coin Flip Conundrum" to choose the best answer to the questions. Describe the role that probability theory played in the brothers' business.

https://disk.yandex.ru/i/DjFvoSWGHuR5UQ

1. How did the Wright brothers originally decide who would fly their airplane first?
A. They had a complicated contest
B. They flipped a coin once
C. Wilbur automatically won the right
D. They flipped coins repeatedly

2. Why does the heads-heads combination take longer to achieve than heads-tails?
A. Because heads appears less frequently than tails
B. Because it requires more than two flips

C. Because it has a move that sends you back to the start
D. Because the probability of heads is lower

3. What example is used to explain why one combination takes longer than the other?
A. A board game comparison with different paths
B. Historical records from the Wright brothers
C. Scientific experiments with special coins
D. Computer simulations of coin flipping

4. According to the mathematical calculations, what is true about the average number of flips needed?
A. Both combinations require the same number of flips
B. Heads-heads requires fewer flips than heads-tails
C. Heads-tails requires fewer flips than heads-heads
D. The number of flips depends on who is flipping

5. What happened when the Wright brothers actually flipped the coin?
A. Orville won the flip and successfully flew the airplane
B. Wilbur won the flip but his flight attempt failed
C. They decided to flip multiple times instead
D. They used the heads-heads method to decide

Activity 158. Reorder the sentences to make a text on mathematical statistics.

a. Another statistic would be the tallest height recorded or the range of heights observed.
b. For example, a medical study might record the heights of 100 children, all age 8.
c. In leisure, many sports fans follow statistical analyses to assess team and player performance.
d. Insurance companies analyze life tables to make inferences and to set insurance rates.
e. It is based on the Latin verb “stare” meaning “to stand.”
f. Making a judgment based on the data that another child outside of the study is of abnormal height would be an example of using data for inferential purposes.
g. Statistics is an indispensable tool used in practically every aspect of life today.
h. Statistics is the branch of mathematics concerned with the methods of collecting, tabulating, and summarizing numerical facts (this is called descriptive statistics), and for making inferences and predictions based on these facts (inferential statistics).
i. Statistics is used extensively in government, business, and commerce to analyze opinion polls, campaign and advertising strategies, business operations, pollution

control, and other environmental concerns, for example, and as well as in scientific research and economic, political, and sociological studies.

j. The average height of the children would be an example of a statistic.

k. The numerical information gathered is called data, and an individual numerical fact about the data is called a statistic.

l. The word "statistic" was coined by the German political scientist Gottfried Achenwall (1719–72) to mean "a summary of how things stand."

m. Weather predictions are based on methods of statistical inference, for example, as are the assessed effectiveness of new drugs, new medical procedures, and other health practices.

Activity 159. Do the quiz on probability theory and mathematical statistics. In pairs, compare your answers.

1. **What prompted the development of probability theory in the 17^{th} century?**
   a. architectural calculations
   b. financial markets
   c. betting and gaming
   d. astrological predictions
2. **Who sought advice from Blaise Pascal about divvying up stakes in interrupted games, leading to the birth of probability theory?**
   a. Isaac Newton
   b. Chevalier de Méré
   c. Pierre de Fermat
   d. Jacob Bernoulli
3. **What is the key principle behind probability theory discussed by Girolamo Cardano and later recognized by Pascal and Fermat?**
   a. law of large numbers
   b. principle of equal likelihood
   c. expected value principle
   d. law of total probability
4. **Which mathematician recognized the wide-ranging applicability of probability beyond gambling, demonstrating its use in medicine and meteorology?**
   a. Pierre-Simon Laplace
   b. Jacob Bernoulli
   c. Carl Friedrich Gauss
   d. John Graunt

5. **What do probability and statistics explore?**
   a. Probability explores unknown collections; statistics explores known samples.
   b. Probability explores known samples; statistics explores unknown collections.
   c. Both explore unknown collections.
   d. Both explore known samples.
6. **Who recognized the repeated appearance of the normal distribution and wrote down a mathematical equation for it in 1733?**
   a. Jacob Bernoulli
   b. Abraham de Moivre
   c. Pierre-Simon Laplace
   d. Siméon Denis Poisson
7. **Which mathematician is considered the most important statistician of the 20th century and transformed statistics into a powerful scientific tool?**
   a. Karl Pearson
   b. John von Neumann
   c. Ronald Aylmer Fisher
   d. William Sealy Gosset
8. **What significant statistical work did Francis Galton contribute in the 1860s?**
   a. inference from birth and death records
   b. analysis of human heredity
   c. development of chi-squared test
   d. introduction of hypothesis testing
9. **Who founded game theory in 1926, recognizing its applications to economics and social sciences?**
   a. Ronald Aylmer Fisher
   b. John Forbes Nash, Jr.
   c. William Sealy Gosset
   d. John von Neumann

Activity 160. Read the article. Review your answers to the quiz in Activity 159.

Questions in betting and gaming provided much of the early impetus for the development of probability theory. In 1654 Chevalier de Méré, a French nobleman with a taste for gambling, wrote a letter to the mathematician Blaise Pascal (1623–62) seeking advice about divvying up stakes from interrupted games.

For example, suppose, in a friendly game of tennis, two players each lay down a stake of $100 in a gamble to win “best out of nine” games, but rain interrupts play after just four games, with one player having won three games, the second only one. What then would be the fair way to divide the $200 pot? Of course, the division of money should somehow reflect each player's likelihood of winning the gamble if the series of games were to be finished.

Pascal communicated the concern of analyzing situations like these to his colleague Pierre de Fermat (1601–65), and their subsequent correspondences on the issue represented the birth of the new field of probability theory. Both mathematicians solved de Méré's "problem of points" (using two entirely different approaches, incidentally) and then later worked together to generalize the problem and extend their analyses to other types of games of chance. Their discoveries aroused the interest of other European scholars. In 1656 the Dutch physicist-astronomer-mathematician Christiaan Huygens (1629–95) published "On Reasoning in Games of Chance" summarizing and extending the ideas developed by Pascal and Fermat. He phrased their work in terms of a new notion, that of expected value. It proved to be very fruitful.

The key principle behind probability theory is the idea that if a situation can be described in terms of possible outcomes that are equally likely, then the probability of any particular outcome occurring is 1 divided by the total number of outcomes. This principle was actually first recognized and discussed more than a century earlier by the Italian mathematician and physician Girolamo Cardano (1501–76) in his work "Book on Games of Chance". This text, however, was not published until 1663, 9 years after Pascal and Fermat had solved de Méré's problem. It is likely that Cardano would be known as "the father of probability theory" had the work been published during his lifetime. Cardano also recognized the law of large numbers.

The Swiss mathematician Jacob Bernoulli (1654–1705) of the famous Bernoulli family recognized the wide-ranging applicability of probability in fields outside of gambling. His book "The Art of Conjecture", published posthumously in 1713, demonstrated the use of the theory in medicine and meteorology. It was also the first comprehensive text dealing with issues of statistics.

In some sense, probability and statistics represent two sides of the same fundamental situation. Probability explores what can be said about an unknown sample of a known collection. (For example, we know all possible numerical combinations from a pair of dice. What then is the most likely outcome from tossing a pair of dice?) Statistics explores what can be said about an unknown collection given a small sample. (If 37 of these 100 people brush their teeth twice a day, what can be said about teeth-brushing habits of the entire population?) The two fields remained closely intertwined during much of the 18th century and the early part of the next century.

In 1733 Abraham de Moivre (1667–1754) recognized the repeated appearance of the normal distribution in scientific studies and wrote down a mathematical equation for it. It first became apparent from the "randomness" of errors in astronomical observations and in scientific experiments.

The latter half of the 19th century saw significant progress in developing and understanding the theoretical foundations of probability theory. This was chiefly due to the work of French mathematicians-astronomers-physicists Joseph-Louis Lagrange (1736–1813) and Pierre-Simon Laplace (1749–1827), German genius Carl Friedrich Gauss (1777–1855), and the French mathematician Siméon Denis Poisson (1781–1840) who, among other things, mathematically proved the law of large numbers. The most important publication in this era on the theory of probability was Laplace's 1812 text "Analytical Theory of Probability". In it, Laplace collected and extended everything known on the subject at that time. Russian mathematicians Pafnuty Chebyshev (1821–94), Andrei Markov (1856–1922), and Alexandr Lyapunov (1857–1918) further developed the mathematical underpinnings of the subject in the late 19th century.

Basic statistical thought can be deemed as having developed considerably earlier. The ancient Egyptians compiled data concerning population and wealth as early as 3050 B.C.E., developing simple techniques to collate and record the numerical information gathered. The ancient Chinese undertook similar studies around 2300 B.C.E. A census was taken in 594 B.C.E. by the Greeks for the purpose of levying taxes, and Athens undertook a population census in 309 B.C.E. The Romans also kept census records, as well as records of births and deaths, and gathered significant quantities of numerical information from geographic surveys taken across the entire empire. Very few statistical records were kept during the period of the Middle Ages, however.

In 1662 John Graunt analyzed birth and death records and produced the first life table. The purpose of the table was to make general observations and predictions about life expectancy for classes of members of a particular population. This work represented a significant step toward analyzing data for the purposes of inference.

In 1790 the United States took its first decennial census, heralding the return of census taking. Several European nations followed suit soon afterward. The Belgian scholar Lambert Adolphe Quételet (1796–1874) analyzed the nation's records and made important observations about the influence of age, gender, occupation, and economic condition on mortality. In 1835 he attempted to apply probabilistic methods to the study of human characteristics, both physical and behavioural. He used them to give what he hoped was a complete description of the "average man." Although Quételet's work was generally highly respected, his attempt to apply it to the field of behavioural science was met with criticism. In the 1860s, the English scholar Francis Galton (1822–1911) attempted to apply statistics methods to the study of human heredity. His work was influential and helped define statistics as a mathematics discipline in its own right.

At the turn of the 20th century, the corporate world began to recognize the relevance and usefulness of statistics, especially in issues of quality control, economics, insurance, and telecommunications. Many large companies began hiring statisticians.

While working for an English brewing company, the industrial scientist William Sealy Gosset (1876–1937) developed the Student's t-test, allowing for the ability to derive reliable information from small samples. (Company policy forbade its employees to publish. Gosset did so in any case, writing under the pseudonym "Student"). The English mathematician Karl Pearson (1857–1936) developed the chi-squared test and is considered the founder of modern hypothesis testing.

Ronald Aylmer Fisher (1890–1962) is considered the most important statistician of the 20th century. His 1925 text "Statistical Methods for Research Workers" transformed statistics into a powerful scientific tool. He clarified many of the mathematical principles on which the discipline is based. Fisher also developed methods of multivariate analysis to properly analyze problems involving more than one variable.

In 1926, the pure and applied mathematician John von Neumann (1903–57) founded game theory — a mathematical framework for analyzing games of chance, such as poker, that involve strategy and choice on the parts of the players. Von Neumann recognized the applications of the theory to economics and social sciences. The work of Nobel Laureate John Forbes Nash, Jr., (1928–2015) took its applications to economics to a profound level.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 161. Identify the individuals based on the descriptions from the text in Activity 160.

Abraham de Moivre / Alexandr Lyapunov / Andrei Markov / Blaise Pascal / Carl Friedrich Gauss / Christiaan Huygens / Francis Galton / Girolamo Cardano / Jacob Bernoulli / John Forbes Nash, Jr. / John Graunt / John von Neumann / Joseph-Louis Lagrange / Karl Pearson / Lambert Adolphe Quételet / Pafnuty Chebyshev / Pierre de Fermat / Pierre-Simon Laplace / Ronald Aylmer Fisher / Siméon Denis Poisson / William Sealy Gosset

1. Analyzed birth and death records in 1662, producing the first life table.
2. A Belgian scholar who, in 1835, attempted to apply probabilistic methods to the study of human characteristics and made important observations about mortality.
3. Considered the most important statistician of the 20th century, transformed statistics into a powerful scientific tool.
4. A Dutch physicist-astronomer-mathematician who, in 1656, published "On Reasoning in Games of Chance," summarizing and extending ideas developed by Pascal and Fermat, introducing the notion of expected value.

5. An English mathematician who developed the chi-squared test and is considered the founder of modern hypothesis testing.
6. An English scholar who, in the 1860s, attempted to apply statistical methods to the study of human heredity.
7. A French mathematician who, in the latter half of the 19th century, contributed to the mathematical underpinnings of probability theory.
8. A French mathematician-astronomer-physicist who, in 1812, published "Analytical Theory of Probability," collecting and extending everything known on the subject at that time.
9. A French mathematician-astronomer-physicist who, in the latter half of the 19th century, contributed to the theoretical foundations of probability theory.
10. A German genius who, in the latter half of the 19th century, contributed to the theoretical foundations of probability theory.
11. An industrial scientist who, while working for an English brewing company, developed the Student’s t-test.
12. An Italian mathematician and physician who, in 1663, posthumously had his work "Book on Games of Chance" published, recognized the law of large numbers.
13. A mathematician who collaborated with Blaise Pascal to solve Chevalier de Méré's "problem of points," contributing to the development of probability theory.
14. A mathematician who received a letter from Chevalier de Méré in 1654, leading to the birth of probability theory in collaboration with Pierre de Fermat.
15. A mathematician who recognized the repeated appearance of the normal distribution in scientific studies in 1733.
16. A Nobel Laureate whose work took game theory applications to economics to a profound level.
17. A pure and applied mathematician who founded game theory in 1926, recognizing applications to economics and social sciences.
18. A Russian mathematician who, in the late 19th century, further developed the mathematical underpinnings of probability theory.
19. A Swiss mathematician who recognized the wide-ranging applicability of probability in fields outside of gambling; his posthumously published book in 1713 demonstrated the use of probability theory in medicine and meteorology.

Activity 162. Rearrange the events in chronological order according to the text of Activity 160. Provide dates where possible.

a. Abraham de Moivre recognizes the normal distribution in scientific studies and writes a mathematical equation for it.

b. Athens conducts a population census.
c. Chevalier de Méré writes to Blaise Pascal seeking advice on divvying up stakes from interrupted games.
d. Christiaan Huygens publishes "On Reasoning in Games of Chance," introducing the notion of expected value.
e. Fisher's text clarifies mathematical principles in statistics.
f. Francis Galton attempts to apply statistical methods to the study of human heredity.
g. Girolamo Cardano's "Book on Games of Chance" is published, recognizing the law of large numbers.
h. Greeks take a census for tax purposes.
i. Jacob Bernoulli's posthumous book "The Art of Conjecture" demonstrates the use of probability theory in medicine and meteorology.
j. John Graunt analyzes birth and death records, producing the first life table.
k. John von Neumann founds game theory.
l. Pierre-Simon Laplace publishes "Analytical Theory of Probability."
m. The corporate world recognizes the relevance and usefulness of statistics, leading to the hiring of statisticians.
n. The United States takes its first decennial census.

Activity 163. Determine whether the statements are true or false by quoting from the text in Activity 160.

1. A substantial number of statistical records were maintained during the Middle Ages.
2. Abraham de Moivre, in 1733, identified the recurrent presence of the normal distribution in scientific studies and formulated a mathematical equation for it.
3. In the early 20th century, businesses started acknowledging the importance and utility of statistics, particularly in quality control, economics, insurance, and telecommunications.
4. Jacob Bernoulli from the renowned Bernoulli family acknowledged the broad applicability of probability beyond gambling.
5. John von Neumann established game theory in 1916.
6. Karl Pearson developed the Student's t-test.
7. Pascal and Fermat's discussions contributed to the birth of mathematical statistics.
8. Probability theory's fundamental principle asserts that if situations have equally likely possible outcomes, the probability of a specific outcome is 1 divided by the total number of outcomes.
9. The development of probability theory was significantly driven by inquiries in betting and gaming.
10. The life table was created without the intention of making general observations and predictions about life expectancy for specific population classes.

Activity 164. In groups, discuss the points. Refer to the text in Activity 160.

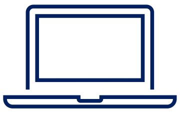
1. How did questions in betting and gaming provide the initial motivation for the development of probability theory, as seen in the interactions between Chevalier de Méré and Blaise Pascal?
2. How did Pascal and Fermat's collaboration, initiated by the concerns raised by de Méré, lead to the birth of probability theory, and what were the key elements of their solutions to the "problem of points"?
3. How did Christiaan Huygens contribute to the development of probability theory, and what role did the notion of expected value play in summarizing and extending the ideas of Pascal and Fermat?
4. What is the key principle behind probability theory, as articulated by Girolamo Cardano, and how did it lay the groundwork for the later work of Pascal and Fermat?
5. How did Jacob Bernoulli demonstrate the wide-ranging applicability of probability in fields beyond gambling, particularly in medicine and meteorology, through his book "The Art of Conjecture"?
6. Explore the relationship between probability and statistics, considering examples such as the known outcomes from a pair of dice and the inference about teeth-brushing habits of a population based on a small sample.
7. Delve into how Abraham de Moivre recognized the significance of the normal distribution, initially observed from errors in astronomical observations, and its mathematical representation.
8. Examine the contributions of Lagrange, Laplace, Gauss, and Poisson in the 19^{th} century, including their mathematical proof of the law of large numbers and Laplace’s pivotal work in the 1812 text “Analytical Theory of Probability.”
9. Trace the historical development of statistics from ancient civilizations to John Graunt's analysis of birth and death records in 1662, highlighting key milestones in the collection and analysis of numerical information.
10. Explore the recognition of statistics in the corporate world during the 20^{th} century, the contributions of statisticians like Gosset, Pearson, and Fisher, and the transformative impact of Fisher’s 1925 text on statistical methods for research workers.

Activity 165. In writing, prove that probability theory and mathematical statistics represent two sides of the same coin. Refer to the text in Activity 160.

"Eureka!" (Archimedes of Syracuse)

Module 5. History of Mathematics

Unit 21. Babylonian Mathematics



Activity 166. Do the quiz on Babylonian mathematics. In pairs, compare your answers.

1. **Where did the Babylonians of 2000 B.C.E. live?**
   a. Greece
   b. Mesopotamia
   c. Egypt
   d. Rome
2. **What material did the Babylonians use for record-keeping?**
   a. papyrus
   b. stone
   c. clay tablets
   d. animal skins
3. **How did Babylonians represent numbers greater than 59?**
   a. additive system
   b. binary system
   c. base-60 place-value system
   d. Roman numerals
4. **Why did Babylonians choose a sexagesimal system?**
   a. It is highly divisible.
   b. It reflects lunar cycles.
   c. It is based on prime numbers.
   d. It aligns with the number of fingers and toes.
5. **What ambiguity arose in the Babylonian numeral system?**
   a. lack of symbols
   b. symbol shapes
   c. no zero symbol
   d. no fraction representation
6. **How did Babylonians solve quadratic equations?**
   a. linear equations
   b. reciprocal tables

c. Pythagorean theorem
d. famous quadratic formula

7. **What method did Babylonians use for cubic equations?**
   a. division
   b. multiplication
   c. setting variables
   d. Pythagorean theorem
8. **What did Babylonians use to compute the diagonal length of a square?**
   a. arithmetic mean
   b. Pythagoras's theorem
   c. trigonometry
   d. reciprocal tables
9. **What evidence suggests Babylonians enjoyed mathematics for its own sake?**
   a. practical applications
   b. Pythagorean triples
   c. tables of powers
   d. linear equations
10. **How did Babylonians approximate the square root of two?**
   a. trial and error
   b. Heron's method
   c. linear interpolation
   d. exponential growth

Activity 167. Read the article. Review your answers to the quiz in Activity 166.

The Babylonians of 2000 B.C.E. lived in Mesopotamia, the fertile plain between the Euphrates and Tigris Rivers in what is now Iraq. We are fortunate that the peoples of this region kept extensive records of their society — and their mathematics — on hardy sun-baked clay tablets. A large number of these tablets survive today. The Babylonians used a simple stylus to make marks in the clay and developed a form of writing based on cuneiform (wedge-shaped) symbols.

The mathematical activity of the Babylonians seems to have been motivated, at first, by the practical everyday needs of running their society. Many problems described in early tablets are concerned with calculating the number of workers needed for building irrigation canals and the total expense of wages, for instance. But many problems described in later texts have no apparent practical application and clearly indicate an interest in pursuing mathematics for its own sake.

The Babylonians used only two symbols to represent numbers: the symbol to represent a unit and the symbol to represent a group of ten. A simple additive system was used to represent the numbers 1 through 59. A base-60 place-value system was then used to represent numbers greater than 59. Spaces were inserted between clusters of symbols.

Historians are not clear as to why the Babylonians chose to work with a sexagesimal system. A popular theory suggests that this number system is based on the observation that there are 365 days in the year. When rounded to the more convenient (highly divisible) value of 360, we have a multiple of 60. Vestiges of this number system remain with us today. For example, we use the number 360 for the number of degrees in a circle, and we count 60 seconds in a minute and 60 minutes per hour.

There were two points of possible confusion with the Babylonian numeral system. With no symbol for zero, it is not clear whether the numeral represents 61 (as one unit of 60 plus a single unit), 3601 (as one unit of 602 plus a single unit), or even 216,060, for instance. Also, the Babylonians were comfortable with fractions and used negative powers of 60 to represent them (just as we use negative powers of 10 to write fractions in decimal notation). But with no notation for the equivalent of a decimal point, the symbol could also be interpreted to mean 1 + (1/60), or (1/60) + (1/602), or even 60 + (1/604), for instance. As the Babylonians never developed a method for resolving such ambiguity, we assume then that it was never considered a problem for scholars of the time. (Historians suggest that the context of the text always made the interpretation of the numeral apparent.)

The Babylonians compiled extensive tables of powers of numbers and their reciprocals, which they used in ingenious ways to perform arithmetic computations. (For instance, a tablet dated from 2000 B.C.E. lists all the squares of the numbers from one to 59, and all the cubes of the numbers from one to 32.) To compute the product of two numbers "a" and "b", Babylonian scholars first computed their sum and their difference, read the squares of those numbers from a table, and divided their difference by four. (In modern notation, this corresponds to the computation: $ab = (1/4)\ [(a + b)^2 – (a – b)^2]$.) To divide a number "a" by "b", scholars computed the product of "a" and the reciprocal 1/b (recorded in a table): ab = a × (1/b). The same table of reciprocals also provided the means to solve linear equations: bx = a. (Multiply "a" by the reciprocal of "b".)

Problems in geometry and the computation of area often lead to the need to solve quadratic equations. For instance, a problem from one tablet asks for the width of a rectangle whose area is 60 and whose length is seven units longer than the width. In modern notation, this amounts to solving the equation $x(x + 7) = x^2 + 7x = 60$. The scribe who wrote the tablet then proffers a solution that is equivalent to the famous quadratic formula: $x = \sqrt{(7/2)^2 + 60} – (7/2) = 5$. (Square roots were computed by examining a table of squares.)

Problems about volume lead to cubic equations, and the Babylonians were adept at solving special equations of the form: $ax^3 + bx^2 = c$. (They solved these by setting n = (ax)/b, from which the equation can be rewritten as $n^3 + n^2 = ca^2/b^3$. By examining a table of values for $n^3 + n^2$, the solution can be deduced.)

It is clear that Babylonian scholars knew of Pythagoras's theorem, although they wrote no general proof of the result. If the width of a rectangle is four units and the length of its diagonal is five units, what is its breadth? Four times four is 16, and five times five is 25. Subtract 16 from 25 and there remains nine. What times what equals nine? Three times three is nine. The breadth is three.

The Babylonians used Pythagoras's theorem to compute the diagonal length of a square, and they found an approximation to the square root of two accurate to five decimal places. (It is believed that they used a method analogous to Heron's method to do this.) Babylonian scholars were also interested in approximating the areas and volumes of various common shapes by using techniques that often invoked Pythagoras's theorem.

Most remarkable is a tablet that lists 15 large Pythagorean triples. As there is no apparent practical need to list these triples, this strongly suggests that the Babylonians did indeed enjoy mathematics for its own sake.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 168. Determine whether the statements are true or false by quoting from the text in Activity 167.

1. Babylonian scholars compiled extensive tables of powers and reciprocals, showcasing sophisticated arithmetic methods, encompassing solving linear, quadratic, and cubic equations.
2. Babylonians developed a systematic approach to resolve ambiguity in numeral interpretation, ensuring precision in their mathematical writings.
3. Babylonians opted for the sexagesimal system due to its mathematical convenience, with historians providing a clear understanding of this choice.
4. Employing intricate cuneiform symbols, the Babylonians had a diverse set of symbols for each number in their numerical system.
5. Initially driven by abstract pursuits, Babylonian mathematical challenges later incorporated practical applications in their texts.
6. The Babylonian numeral system lacked a zero symbol, leading to ambiguity, and they utilized negative powers of 60 for fractions with no decimal point.
7. The Babylonians, residing in Mesopotamia circa 2000 B.C.E., meticulously recorded their society and mathematical endeavours on robust clay tablets.
8. Utilizing a base-10 place-value system, Babylonians employed symbols for all numbers between 1 and 59, simplifying their numerical representation.

Activity 169. In groups, discuss the points. Refer to the text in Activity 167.

1. Discuss the shift in the Babylonians' mathematical focus from practical everyday needs, such as building irrigation canals, to abstract pursuits for the sake of mathematics itself. What might have prompted this transition?
2. Explore the simplicity of the Babylonian numeral system with only two symbols. How might such a system, based on units and groups of ten, compare to or differ from modern numerical representations?
3. Delve into the theories surrounding the Babylonians' choice of the sexagesimal system. How did the observation of 365 days in a year influence their numerical system, and what vestiges of this system are still present in contemporary measurements?
4. Discuss the challenges posed by the Babylonian numeral system, particularly the absence of zero and a decimal point. How might this ambiguity have impacted their mathematical calculations, and why do historians believe it wasn't a concern for Babylonian scholars?
5. Examine the Babylonians' use of tables listing powers of numbers and their reciprocals. How did this approach aid them in performing arithmetic computations, and how does it compare to modern computational methods?
6. Explore the Babylonians' methods for solving quadratic and cubic equations. How do their approaches, such as using squares and cubes tables, compare to contemporary methods for solving similar equations?
7. Discuss the Babylonians' application of Pythagoras's theorem in practical problem-solving, such as finding the breadth of a rectangle. How did they employ this theorem in areas beyond geometry, like computing square roots?
8. Explore how the Babylonians approximated the square root of two and applied Pythagoras's theorem to various shapes. How did these approximations demonstrate their interest in real-world measurements and calculations?
9. Investigate the significance of the tablet listing 15 large Pythagorean triples. Why might the Babylonians have documented these triples, and what does it reveal about their enjoying mathematics for its own sake?
10. Discuss the lasting impact of Babylonian mathematics on contemporary mathematical practices. In what ways have their methods influenced modern mathematical systems and approaches?

Activity 170. Write an overview of the text in Activity 167 using Appendix II.

Unit 22. Egyptian Mathematics

Activity 171. Do the quiz on Egyptian mathematics. In pairs, compare your answers.

1. **Where does our knowledge of ancient Egyptian mathematics primarily come from?**
   a. Rosetta Stone
   b. Rhind papyrus
   c. pyramid inscriptions
   d. Library of Alexandria
2. **What was the Egyptian approach to denoting numerals like?**
   a. place-value system
   b. Roman numeral system
   c. hieroglyphic symbols
   d. decimal system
3. **How did Egyptians perform arithmetic calculations without a place-value system?**
   a. mental calculations
   b. using an abacus-like calculating board
   c. with pen and paper
   d. utilizing complex algorithms
4. **What method did the ancient Egyptians use for multiplication?**
   a. long division
   b. successive doubling
   c. Egyptian addition
   d. false position
5. **How did Egyptians express fractions in the Ahmes papyrus?**
   a. with numerators and denominators
   b. using a dot over the number
   c. through symbols only
   d. writing fractions in words
6. **What term is used today for fractions with unit numerators?**
   a. Pythagorean fractions
   b. Hieratic fractions
   c. Greek fractions
   d. Egyptian fractions

7. **What method did the Egyptians employ to solve linear equations, as described in the Ahmes papyrus?**
    a. trial and error
    b. calculus
    c. false position
    d. matrix algebra
8. **What kind of problems make up the majority of the Ahmes papyrus?**
    a. algebraic problems
    b. number theoretic problems
    c. practical problems related to area, volume, and more
    d. geometric proofs
9. **Which problem in the Ahmes papyrus reflects a delight in mathematical thinking for its own sake?**
    a. problem 1: numerical notation
    b. problem 24: false position
    c. problem 50: multiplication techniques
    d. problem 79: houses, cats, mice, and grain

Activity 172. Read the article. Review your answers to the quiz in Activity 171.

Our knowledge of ancient Egyptian mathematics from around 2000 B.C.E. comes chiefly from the Rhind papyrus (Ahmes papyrus). There we learn, for example, that the Egyptians followed a very natural system for denoting numerals: 1 was a vertical stroke |, 2 was two of them ||, 3 was |||, and 4 was ||||, and separate symbols were used for 5, 6, 7, 8, and 9, and for 10, 20, ..., 100, 200, ..., 1000, and so on. All other numbers were represented as groups of these symbols, usually arranged in order from largest to smallest. Like the Roman numeral system, the Egyptian system did not use a place-value system (the symbol for 5, for example, denoted "5" no matter where it appeared in the number). It is very difficult to do pencil-and-paper calculations without place-value notation, but the Egyptians always used a calculating board, much like an abacus, to perform arithmetic calculations, and needed only to record the results. They were therefore not hindered by their cumbersome numerical system. The ancient Egyptians were adept at multiplication, using a method of successive doubling to calculate products. This method is today called Egyptian multiplication.

Division problems lead to fractions. It did not occur to the ancient Egyptians to express fractions with numerators and denominators. In the Rhind papyrus, the mathematician Ahmes simply placed a dot over a number to indicate its reciprocal, except in the case of the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{1}{4}$, each of which had its own symbol. Thus, the Egyptians only dealt with fractions of the form $\frac{1}{n}$ (with the exception of two-thirds). Fractions with unit numerators are known today as Egyptian fractions. All other fractional quantities were expressed as sums of distinct Egyptian fractions. The Egyptian's ability to compute such expressions is impressive. The Rhind papyrus provides reference lists of such expressions, and the first 23 problems in the document are exercises in working with such fractional representations.

The ancient Egyptians were adept at solving linear equations. They used a method called false position to attain solutions. This involves guessing an answer, observing the outcome from the guess, and adjusting the guess accordingly. As an example, problem 24 of the Rhind papyrus asks:

Find the quantity so that when 1/7 of itself is added to it, the total is 19.

To demonstrate the solution, the author suggests a guess of 7. That plus its one-seventh is 8, by far too small, but multiplying the outcome by 19/8 produces the answer of 19 that we need. Thus, 7 × (19/8) must be the quantity we desire.

The majority of problems in the Rhind papyrus are practical in nature, dealing with issues of area (of rectangles, trapezoids, triangles, circles), volume (of cylinders, for example), slopes and altitudes of pyramids (which were built 1,000 years before the text was written), and number theoretic problems about sharing goods under certain constraints. Some problems, however, indicate a delight in mathematical thinking for its own sake. For example, problem 79 asks:

If there are seven houses, each house with seven cats, seven mice for each cat, seven ears of grain for each mouse, and each ear of grain would produce seven measures of grain if planted, how many items are there altogether?

This problem appears in Fibonacci's "The Book of the Abacus", written 600 years before the Rhind papyrus was discovered. A version of this problem also appears as a familiar nursery-rhyme and riddle, "As I Was Going to St. Ives."

(by James Tanton, from Encyclopedia of Mathematics)

Table 29

Rhind Papyrus	**Nursery Rhyme**
If there are seven houses, each house with seven cats, seven mice for each cat, seven ears of grain for each mouse, and each ear of grain would produce seven measures of grain if planted, how many items are there altogether?	As I was going to St. Ives, I met a man with seven wives, Each wife had seven sacks, Each sack had seven cats, Each cat had seven kits: Kits, cats, sacks, and wives, How many were there going to St. Ives?

Activity 173. Determine whether the statements are true or false by quoting from the text in Activity 172.

1. Ancient Egyptians excelled in multiplication, employing a technique of successive doubling for calculations.
2. Ancient Egyptians were aware of expressing fractions with numerators and denominators.
3. Ancient Egyptians were skilled in solving quadratic equations through the method of false position.
4. Egyptians did not always rely on a calculating board or abacus for arithmetic calculations.
5. Egyptians used an intuitive system to represent numerals.
6. Most problems in the Rhind papyrus are practical, involving concepts like area, volume, and the characteristics of pyramids.
7. Not all fractional quantities were represented as sums of distinct Egyptian fractions.
8. The primary source of information on ancient Egyptian mathematics around 2000 B.C.E. is the Rhind papyrus, also known as the Ahmes papyrus.
9. The term "Egyptian fractions" refers to fractions with unit denominators.

Activity 174. In groups, discuss the points. Refer to the text in Activity 172.

1. Discuss the unique system of numerical notation used by the ancient Egyptians, highlighting the symbolic representation of numbers and the absence of a place-value system.
2. Explore the significance of calculating boards, similar to abacuses, in Egyptian mathematics. Discuss how these tools allowed efficient arithmetic calculations despite the limitations of their numerical system.
3. Examine the method of successive doubling employed by the Egyptians for multiplication. Discuss the implications of this technique and its relevance in contemporary mathematics.
4. Delve into the Egyptian approach to fractions, emphasizing the use of symbols and the specific treatment of fractions like 1/2, 1/3, 2/3, and 1/4. Discuss the concept of Egyptian fractions and their representation.
5. Explore the method of false position used by the ancient Egyptians to solve linear equations. Discuss the problem-solving approach involving guesswork and adjustments, using a specific example from the Rhind papyrus.
6. Analyze the majority of problems in the Rhind papyrus, which have practical applications in geometry, volume calculations, and number theory. Discuss the real-world scenarios covered in these mathematical problems.

7. Highlight the impressive computational skills of the ancient Egyptians, as evidenced by their ability to compute complex expressions and their reliance on reference lists for problem-solving.
8. Explore instances in the Rhind papyrus where mathematical problems indicate a delight in mathematical thinking for its own sake. Discuss the significance of such problems in understanding the mathematical mindset of the ancient Egyptians.
9. Compare the mathematical problems in the Rhind papyrus with similar problems found in other mathematical texts, such as Fibonacci's "The Book of the Abacus." Discuss the cross-cultural aspects of mathematical problem-solving.
10. Discuss the enduring legacy of Egyptian mathematics, considering its influence on subsequent mathematical developments and its relevance in modern mathematical discourse.

Activity 175. Write an overview of the text in Activity 172 using Appendix II.

Unit 23. Greek Mathematics

Figure 9. "The School of Athens" (by Raphael)

Activity 176. Look at the image in Figure 9. In pairs, discuss the questions.

1. Who was Raphael?
2. What scholars and scientists are depicted in the painting? What are they famous for?
3. How does the fresco reflect the values of the Renaissance?

Activity 177. Study Table 30. Illustrate how different letters of the Greek alphabet are used in mathematics.

Table 30. Greek Alphabet

№	Letter	Name		№	Letter	Name	
1	Α α	alpha	/ˈælfə/	13	Ν ν	nu	/njuː/
2	Β β	beta	/ˈbiːtə/ /ˈbeɪtə/ (Am)	14	Ξ ξ	xi	/zaɪ/ /ksaɪ/
3	Γ γ	gamma	/ˈɡæmə/	15	Ο ο	omicron	/ˈɒmɪkrɒn/ /oʊˈmaɪkrɒn/ (Br)
4	Δ δ	delta	/ˈdɛltə/	16	Π π	pi	/paɪ/
5	Ε ε	epsilon	/ˈɛpsɪlɒn/ /ɛpˈsaɪlən/ (Br)	17	Ρ ρ	rho	/roʊ/
6	Ζ ζ	zeta	/ˈziːtə/ /ˈzeɪtə/ (Am)	18	Σ σ/ς Ϲ ϲ	sigma	/ˈsɪɡmə/
7	Η η	eta	/ˈiːtə/ /ˈeɪtə/ (Am)	19	Τ τ	tau	/taʊ/ /tɔː/
8	Θ θ	theta	/ˈθiːtə/ /ˈθeɪtə/ (Am)	20	Υ υ	upsilon	/juːpˈsaɪlən/ /ˈʊpsɪlɒn/ /ʌpˈsaɪlən/ (Br) /ˈʌpsɪlɒn/ (Am)
9	Ι ι	iota	/aɪˈoʊtə/	21	Φ φ	phi	/faɪ/
10	Κ κ	kappa	/ˈkæpə/	22	Χ χ	chi	/kaɪ/
11	Λ λ	lambda	/ˈlæmdə/	23	Ψ ψ	psi	/saɪ/ /psaɪ/
12	Μ μ	mu	/mjuː/ /muː/ (Am)	24	Ω ω	omega	/ˈoʊmɪɡə/ (Br) /oʊˈmeɪɡə/ (Am)

Activity 178. Do the quiz on Greek mathematics. In pairs, compare your answers.

1. **Who is considered the first Greek mathematician of note?**
   a. Euclid
   b. Thales of Miletus
   c. Archimedes

d. Pythagoras

2. What did Euclid's work "The Elements" primarily focus on?

a. conic sections
b. trigonometry
c. deductive reasoning and logical proofs
d. squaring the circle

3. In Greek mathematics, how was a "number" conceptualized?

a. a line segment
b. a point
c. a circle
d. an angle

4. What did Plato use as an example of something that cannot be discovered by the senses but can be discovered by logical reasoning?

a. calculus
b. geometry
c. mathematics
d. trigonometry

5. Who is considered the most influential mathematics scholar of all time in Greek mathematics?

a. Archimedes
b. Plato
c. Euclid
d. Pythagoras

6. What did Archimedes of Syracuse solve, and what method did he use?

a. squaring the circle, using a straightedge and compass
b. trisecting an angle, using an Archimedean spiral
c. squaring the parabola, using a straightedge and compass
d. duplicating the cube, using a conic section

7. Who continued work on conic sections and properly defined an ellipse, a hyperbola, and a parabola?

a. Hipparchus
b. Diophantus
c. Ptolemy
d. Apollonius of Perga

8. What did Claudius Ptolemy attempt to prove in the 2nd century C.E.?

a. Euclid's parallel postulate
b. Pythagoras's theorem
c. Archimedes's principle
d. Hipparchus's table

Activity 179. Read the article. Review your answers to the quiz in Activity 178.

The ancient Greeks of ca. 600 B.C.E. to ca. 480 C.E. set the current standards of logical rigour in mathematics. Although many ancient cultures practiced and developed mathematics, it was the Greeks who developed the explicit art of "proof" and explored the power of pure deductive reasoning to its fullest.

We should mention that when speaking of "Greek mathematics," historians include any mathematician who wrote in the Greek language and followed the Greek tradition of mathematical thought. Greek was the common language of the Mediterranean world during ancient times, and many intellectuals from different parts of that region are today considered Greek scholars. For instance, the great Archimedes was from Syracuse, now a part of Italy, and Euclid (ca. 300–260 B.C.E.) is believed to have lived in Alexandria, Egypt.

There are very few original records of Greek work. Initially, knowledge was transmitted only orally from teacher to student. Around 450 B.C.E. the Greeks adopted the ancient Egyptian practice of writing on papyrus scrolls. Unfortunately, papyrus — a grasslike plant grown in the Nile Delta region — decays rapidly away from the exceptionally dry climate of Egypt. The Greeks combated this problem by repeatedly making copies of their works but, because of the effort involved, copied only those pieces they deemed of utmost importance. The first mathematical work preserved and honoured this way was Euclid's masterpiece "The Elements" of ca. 300 B.C.E. Historians have had to rely on commentary made by later scholars to deduce what was accomplished mathematically before the time of Euclid.

Greek scholars approached all of mathematics through the study of geometry. Even their work on the properties of whole numbers, ratios, and proportions, as well as mechanics and astronomy was done in a geometric style. A "number," for instance, was literally a line segment, and a "ratio" was understood in terms of commensurable segments. It is interesting to note that Greek scholars took careful steps to avoid speaking directly of the infinite. (The 5^{th}-century B.C.E. paradoxes on the nature of motion and the infinitely small developed by Zeno of Elea deeply affected Greek thinking.) For instance, Euclid stated that any line segment could be extended to any arbitrary length, but never spoke of lines that were infinitely long. In Euclid's proof of the infinitude of primes, Euclid stated that from any finite list of prime numbers one can always construct one more, but never spoke of the set of primes as infinite.

Many historians regard Thales of Miletus (ca. 625–547 B.C.E.) as the first Greek mathematician of note. Commentaries suggest that Thales identified, and proved, seven key geometric propositions, including that the base angles of an isosceles triangle are always equal and that the inscribed angle from the diameter of a circle is always a right angle, for instance. The great scholar and mystic Pythagoras lived a century later, and he and his followers are credited with the discovery of the famous result about right triangles (today called Pythagoras's theorem) and the discovery of irrational numbers. A great deal of mystery surrounds the life and legend of Pythagoras. He founded a semireligious sect called the Pythagorean Brotherhood (women were equal members) based on certain mystic significances ascribed to whole numbers and their ratios.

The great philosopher Plato (428–348 B.C.E.) wrote a great deal about mathematics in his famous dialogues, demonstrating a deep personal respect for the subject. The five regular polyhedra — the Platonic solids — are named in his honour. In his philosophical treatises, Plato used the example of mathematics as something that cannot be discovered by the senses but can nonetheless be discovered by the power of logical reasoning. He also believed mathematics to be an essential part of a cultured person's education. Philosopher Aristotle (384–322 B.C.E.) adopted the same view and used mathematics as examples in his development of formal logic and his analysis of arguments.

Today, the Greek scholar Euclid is considered to be the most influential mathematics scholar of all time. In his famous work "The Elements", Euclid collated all mathematical knowledge known at his time into a single tome. Although an impressive feat, it was the organization of the text that had the greatest impact. Beginning with a small collection of "self-evident truths," Euclid showed that all mathematical knowledge of his time could be deduced by pure logical reasoning alone. This work demonstrated the power of the mind and set the model for all mathematical research in the future. Mathematicians today still work to the standards of rigour as set by Euclid. Next to the Bible, Euclid's "The Elements" is the most widely published book of all time.

After producing "The Elements", Euclid continued work on the conic sections, on optics, and on general problems in geometry. He continued interest in constructible numbers and no doubt contemplated the classic Greek problem of squaring the circle. (In "The Elements" Euclid had demonstrated general procedures for squaring arbitrary polygonal figures.) This challenge, as well as the problems of trisecting an angle and duplicating the cube, spurred a great deal of significant further research in mathematics for centuries to come.

Archimedes of Syracuse (ca. 287–212 B.C.E.) solved the problem of squaring the parabola, as well as made significant advances in computing the areas and volumes of curved figures and solids. (He also "solved" the problem of squaring the circle by making use of his Archimedean spiral. Unfortunately, his method went beyond the use of a straightedge and compass alone, and so is not a permissible solution to the original problem.)

Apollonius of Perga (ca. 262–190 B.C.E.) continued work on conic sections and is credited for properly defining an ellipse, a hyperbola, and a parabola. Around the same time, Greek astronomer Hipparchus wrote a table of "chord values" (the equivalent to a modern table of sine values), which he used to solve astronomical problems. This represented the beginning development of trigonometry in Greek mathematics, but also marked an end of fervent mathematical development in the Greek tradition. For the five centuries that followed, new developments were limited to straightforward advances in astronomy, trigonometry, and algebra, with just a few notable exceptions.

In the 2^{nd} century C.E., The Greek astronomer Claudius Ptolemy corrected and extended Hipparchus's table and clarified the mathematics that is used to produce such a table. He is also known as one of the first scholars to make a serious attempt at proving Euclid's parallel postulate. In the 3^{rd} century, Diophantus of Alexandria produced his famous text "Arithmetic", from which the study of Diophantine equations was born. In the mid-4^{th} century, the enthusiastic Pappus of Alexandria attempted to revive interest in ardent mathematical research of the Greek style. He produced his treatise "Synagoge" (Collection) to act as a commentary and guide to all the geometric works of his time and included in it a significant number of original results, extensions of ideas, and innovative shifts of perspective. Unfortunately, he did not succeed in his general goal. After Pappus, of note is Hypatia of Alexandria (370–415), the first woman to be named in the history of mathematics, credited for writing insightful commentaries on the works of Apollonius and Diophantus, and Proclus (ca. 410–485), who is noted for his detailed commentary on the work of Euclid and his own attempt to prove the parallel postulate.

The beginning of the 5^{th} century marks a clear end to the tradition of Greek mathematics.

(by James Tanton, from Encyclopedia of Mathematics)

Table 31. Regular Polyhedra

Platonic Solid	tetrahedron	cube	octahedron	dodecahedron	icosahedron
Faces	4	6	8	12	20

Activity 180. Identify the individuals based on the descriptions from the text in Activity 179.

Archimedes / Aristotle / Claudius Ptolemy / Diophantus of Alexandria / Euclid / Hypatia of Alexandria / Pappus of Alexandria / Plato / Proclus / Pythagoras / Thales of Miletus

1. An Enthusiastic mathematician of the mid-4^{th} century who attempted to revive interest in mathematical research, produced the treatise "Synagoge" (Collection).

2. First Greek mathematician of note, identified and proved seven key geometric propositions.
3. First woman named in the history of mathematics, credited for insightful commentaries on the works of Apollonius and Diophantus.
4. Great mathematician from Syracuse, now part of Italy.
5. Great philosopher who wrote extensively about mathematics in his famous dialogues, honoured with the naming of the five regular polyhedra.
6. Greek astronomer of the 2nd century C.E., corrected and extended Hipparchus's table, made a serious attempt at proving Euclid's parallel postulate.
7. Greek scholar believed to have lived in Alexandria, Egypt, considered the most influential mathematics scholar of all time.
8. Mathematician noted for a detailed commentary on the work of Euclid and his attempt to prove the parallel postulate.
9. Mathematician of the 3rd century, produced the famous text "Arithmetic," from which the study of Diophantine equations was born.
10. Mystic and scholar credited with the discovery of Pythagoras's theorem and irrational numbers, founded the Pythagorean Brotherhood.
11. Philosopher who used mathematics as examples in the development of formal logic and analysis of arguments.

Activity 181. Rearrange the events in chronological order according to the text of Activity 179. Provide dates where possible.

a. Apollonius continues work on conic sections, properly defining an ellipse, hyperbola, and parabola.
b. Archimedes solves the problem of squaring the parabola and makes significant advances in computing the areas and volumes of curved figures and solids.
c. Aristotle adopts Plato's view on mathematics and incorporates it into his development of formal logic and analysis of arguments.
d. Diophantus produces his famous text "Arithmetic," giving birth to the study of Diophantine equations.
e. Euclid compiles "The Elements," organizing all mathematical knowledge of his time into a single tome.
f. Greek mathematics experiences a decline, marking the end of the tradition that set the standards for logical rigour and deductive reasoning.
g. Hypatia becomes the first woman named in the history of mathematics, credited for insightful commentaries on the works of Apollonius and Diophantus.

h. Pappus attempts to revive interest in Greek-style mathematical research with his treatise "Synagoge" serving as a commentary and guide to geometric works, including original results and innovative perspectives.
i. Plato writes extensively about mathematics in his dialogues, introduces the concept of mathematics as something discoverable through logical reasoning, not the senses.
j. Proclus provides a detailed commentary on Euclid's work and attempts to prove the parallel postulate.
k. Ptolemy corrects and extends Hipparchus's table of "chord values.", makes a serious attempt to prove Euclid's parallel postulate, contributing to the understanding of geometry.
l. Pythagoras and his followers discover Pythagoras's theorem and irrational numbers.
m. Thales identifies and proves key geometric propositions, lays the foundation for Greek mathematical exploration.

Activity 182. Determine whether the statements are true or false by quoting from the text in Activity 179.

1. Archimedes of Syracuse solved the problem of squaring the circle using only a protractor in adherence to classical Greek geometric methods.
2. Aristotle, the philosopher, expressed a profound respect for mathematics in his dialogues and considered it a vital component of a cultured education.
3. Claudius Ptolemy and his followers are credited with the famous result concerning right triangles.
4. Euclid, in his proof of the infinitude of primes, explicitly mentioned the set of primes as infinite.
5. Euclid's "The Elements" stands as the most influential mathematical work, organizing knowledge and establishing enduring standards of rigour.
6. Historians classify as Greek mathematicians those who wrote in Greek and followed the Greek mathematical tradition, regardless of their geographic origins.
7. Thales of Miletus is often recognized as the earliest noteworthy Greek mathematician, acknowledged for proving essential geometric propositions.
8. The Greeks adopted the practice of writing on papyrus scrolls around 450 B.C.E., a departure from the ancient Egyptian method.
9. The Greeks significantly shaped the logical rigour in mathematics from around 600 B.C.E. to 480 C.E.

Activity 183. In groups, discuss the points. Refer to the text in Activity 179.

1. Explore the fundamental contributions of ancient Greek mathematicians, such as Euclid, to geometry. Discuss the development of Euclidean geometry and its lasting impact on mathematical thinking.
2. Delve into the Pythagorean theorem and its origins within the Pythagorean school. Discuss its applications, proofs, and significance in both geometry and mathematics as a whole.
3. Examine the work of Archimedes, particularly in the field of mathematical physics. Discuss Archimedes' principle, the law of the lever, and his methods for calculating areas and volumes.
4. Discuss the contributions of ancient Greek mathematicians, such as Zeno and Eudoxus, to the understanding of infinite series and calculus precursors. Explore the challenges they faced in dealing with the concept of infinity.
5. Explore the philosophical aspects of mathematics in ancient Greece, particularly the views of Plato and Aristotle. Discuss how their philosophical ideas influenced the development of mathematical thought.
6. Discuss the advancements in mathematics during the Hellenistic period, including contributions from mathematicians like Apollonius of Perga. Explore the developments in conic sections and other geometric theories.
7. Examine the mathematical notations used by ancient Greek mathematicians. Discuss the symbols, methods of representation, and how these notations evolved over time.
8. Explore the role of mathematics in ancient Greek education. Discuss how mathematical concepts were taught, the significance of mathematics in the curriculum, and its impact on intellectual development.
9. Investigate the contributions of ancient Greek mathematicians to number theory: the study of prime numbers, perfect numbers, and the sieve of Eratosthenes.

Activity 184. Write an overview of the text in Activity 179 using Appendix II.

Unit 24. Indian Mathematics

Activity 185. Do the quiz on Indian mathematics. In pairs, compare your answers.

1. **What was the significant Indian invention that profoundly influenced Western mathematics?**
   a. Egyptian numeral system
   b. number zero
   c. place-value decimal system
   d. Roman numeral system
2. **How many symbols were used in the Indian place-value system to represent numbers?**
   a. 5
   b. 7
   c. 10
   d. 12
3. **What was the basic unit of length used by the Indus civilization in their decimal system?**
   a. Indus meter
   b. Roman inch
   c. Indus inch
   d. Egyptian cubit
4. **During which period was mathematics in India driven by the needs of Jainism, focusing on accurate astronomical observations?**
   a. Vedic period
   b. Maurya period
   c. Gupta period
   d. Jainism period
5. **Who was the mathematician credited with developing a theory of trigonometry and evaluating π with high accuracy?**
   a. Aryabhata
   b. Bhaskara II
   c. Brahmagupta
   d. Varahamihira
6. **In which century was the Bakhshali manuscript discovered, revealing Indian mathematicians' comfort with fractions and algebraic manipulations?**

a. the 5^{th} century
b. the 7^{th} century
c. the 12^{th} century
d. the 14^{th} century

7. Who was the mathematician who gave zero the status of a number?

a. Aryabhata
b. Bhaskara II
c. Brahmagupta
d. Varahamihira

8. Which 14^{th}-century scholar made significant advances in analysis, including producing infinite series expansions, and discovering the binomial theorem?

a. Aryabhata
b. Bhaskara II
c. Madhava
d. Varahamihira

9. When did the decimal system spread to the Islamic world and eventually reach Europe?

a. the 7^{th} century
b. the 9^{th} century
c. the 12^{th} century
d. the 14^{th} century

10. What role did Arab scholars play in the transmission of Indian mathematical knowledge to the West?

a. They ignored Indian contributions.
b. They actively preserved and translated Indian texts.
c. They opposed the decimal system.
d. They developed their own numerical system.

Activity 186. Read the article. Review your answers to the quiz in Activity 185.

The entire course of Western mathematics was profoundly affected by a single Indian invention, that of the place-value decimal system. That every possible number can be expressed via a set of just 10 symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, by making careful use of the place of each symbol, seems such a simple notion nowadays that it is hard to appreciate its profound importance. This elegant notation system provided the means for Indian scholars to perform complicated arithmetical computations with relative ease, which in turn led to significant developments in numerical techniques, approximation methods, and the theory of arithmetic. Only when other cultures adopted the place-value decimal system from India could they accomplish the same mathematical feats that this culture had already developed.

The earliest dated evidence of mathematical activity in the Indian subcontinent goes back to the Indus civilization of 2500 B.C.E. Bronze weights and graded rods (rulers) from the period show that these people were already working with a decimal system. The Indus people worked with a basic unit of length 1.32 in. long (today called the "Indus inch"), 10 of which make their version of a "foot." Excavations show that the weights and graded rods were used extensively in construction.

The earliest written records of Indian culture are the religious texts the Vedas, dating between 1500 B.C.E. and 800 B.C.E. Although not mathematical in content, appendices to the texts give specific rules for constructing altars, exhibiting a thorough understanding of the basic principles of geometry. Early versions of the digits 0 through 9 were used at this time.

By 600 C.E., the Vedic religion had gone into decline, and Jainism came to the fore. During this period mathematics was driven by the needs of the religion and its demands for careful astronomical observations to pinpoint the exact times of religious observances and the development of an accurate calendar. The decimal representation system was now fully developed, and scholars were able to make precise and surprisingly accurate calculations. The mathematician Aryabhata (ca. 500 C.E.), for instance, had developed a theory of trigonometry to aid astronomical calculations, had developed methods for extracting square roots, evaluated π to a high degree of accuracy, and was able to find integer solutions to a large class of equations that arose from astronomical theories.

One written text from this period was discovered in 1881 in the town of Bakhshali, now in Pakistan. Written on birch bark, the Bakhshali manuscript shows that mathematicians were also comfortable with fractions, basic algebraic manipulations (they used a dot to represent an unknown quantity), and sophisticated approximation formulae.

Two mathematical research centres were formed in India during the era of Jainism, both astronomical observatories. Aryabhata headed the first centre at Kusumapura in the northeast of the Indian subcontinent, and the mathematician Varahamihira, who also made contributions to astronomy and trigonometry, headed the second centre at Ujjain, also in the north.

Varahamihira was succeeded by the 7th-century mathematician Brahmagupta, who, in his famous work "The Opening of the Universe", introduced and explained the arithmetic of non-positive numbers. He was the first mathematician in history to give zero the status of a number, defining it to be the result of subtracting a quantity from itself. Brahmagupta's work also includes a formula for the area of a cyclic quadrilateral in terms of its sides (today called Brahmagupta's formula), and presents methods for solving linear and quadratic equations, as well as systems of equations. Brahmagupta also developed sophisticated interpolation techniques for computing sine values in trigonometry.

For the next 200 years, Indian scholars worked to refine further methods of trigonometry and techniques of astronomical calculation. The mathematician Bhaskara II (Bhaskaracharya) of the 12th century made advances in number theory, algebra, combinatorics, and astronomy, and wrote a comprehensive text summarizing the state of mathematics and astronomy in India at his time. Soon afterward, other Indian scholars developed these ideas further. Jaina mathematicians also clarified the standard exponent rules and manipulated exponents in a manner that suggests today that they were also familiar with the basic principles of logarithms.

The 14th-century scholar Madhava of Sangamagramma made significant advances in analysis. He produced the infinite series expansions of trigonometric and inverse trigonometric functions (today called Taylor series), discovered the binomial theorem, and even produced Gregory's series for π, which he used to approximate its value to a considerably accurate degree.

During the first millennium India had very little contact with the cultures of the West. News of the decimal representation system, however, did manage to spread to other countries relatively quickly. A manuscript written in Syria in 662 discusses the new method of calculation, and there is evidence that the decimal system was being used in Cambodia and other Asian countries soon afterward. By the 9^{th} century, the decimal system was in common use in the Islamic world, and from there it was quickly transmitted to Europe. Arab scholars maintained a keen interest in the work of Indian mathematicians for the centuries that followed and took an active role in preserving and translating many Indian texts.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 187. Identify the individuals based on the descriptions from the text in Activity 186.

Aryabhata / Bhaskara II (Bhaskaracharya) / Brahmagupta / Madhava of Sangamagramma / Varahamihira

1. A 12^{th}-century mathematician who advanced number theory, algebra, combinatorics, and astronomy, contributing to the state of Indian mathematics.
2. Mathematician and astronomer who succeeded Aryabhata, contributing to astronomy and trigonometry, and heading an astronomical observatory in Ujjain.
3. Mathematician who developed a theory of trigonometry, methods for extracting square roots, and made precise astronomical calculations.
4. Mathematician who introduced zero as a number, explained the arithmetic of non-positive numbers, and developed interpolation techniques for computing sine values.
5. Scholar who made significant advances in analysis, producing infinite series expansions and contributing to the understanding of trigonometric functions.

Activity 188. Rearrange the events in chronological order according to the text of Activity 186. Provide dates where possible.

a. A manuscript in Syria discusses the decimal system, suggesting its spread beyond India.
b. Jaina mathematicians clarify exponent rules and manipulate exponents, possibly hinting at familiarity with logarithmic principles.
c. Madhava of Sangamagramma makes significant advances in analysis, producing infinite series expansions, discovering the binomial theorem, and approximating π with Gregory's series.
d. Mathematics becomes crucial for astronomical observations and calendar development.
e. The Bakhshali manuscript is discovered in Pakistan, revealing mathematicians' comfort with fractions, algebraic manipulations, and sophisticated approximation formulae.
f. The decimal system becomes common in the Islamic world, later transmitted to Europe.
g. The Indus civilization exhibits the earliest evidence of mathematical activity, using a decimal system with bronze weights and graded rods.
h. The Vedas, the earliest written records of Indian culture, provide appendices with geometric principles, showcasing early versions of the digits 0 through 9.

Activity 189. Determine whether the statements are true or false by quoting from the text in Activity 186.

1. Although not directly mathematical, the Vedas, the earliest written records of Indian culture, include guidelines for constructing altars that reveal a deep understanding of fundamental geometric principles.
2. Brahmagupta, a mathematician from the 7^{th} century, established zero as a number, explored non-positive numbers, and presented formulas for areas and solving equations.
3. Jainism had no impact on the progress of mathematics in ancient India, and scholars primarily concentrated on religious practices rather than mathematical developments.
4. The decimal system's origins can be traced back to the Vedic period, and the Vedas extensively cover mathematical content.
5. The decimal system's usage can be traced back to the Indus civilization around 2500 B.C.E., as suggested by their utilization of a decimal system in bronze weights and graded rods.

6. The decline of Jainism and the emergence of Vedic religion prompted mathematical developments in India, particularly in accurate astronomical observations and calendar construction.
7. The dissemination of the decimal system to other nations did not occur until the 17th century.
8. The Western mathematical tradition was significantly impacted by the Indian invention of the place-value decimal system, enabling the representation of any number with just 10 symbols.

Activity 190. In groups, discuss the points. Refer to the text in Activity 186.

1. Explore the profound impact of the Indian invention of the place-value decimal system on the entire course of Western mathematics.
2. Discuss the evidence of mathematical activity in the Indus civilization, including the use of a decimal system and its application in construction.
3. Analyze the mathematical content in the appendices of the Vedas, showcasing rules for constructing altars and a grasp of basic geometric principles.
4. Examine how the decline of Vedic religion and the rise of Jainism influenced mathematics, driven by astronomical needs, resulting in precise calculations and advancements by mathematician Aryabhata.
5. Explore the significance of the Bakhshali manuscript, revealing mathematicians' comfort with fractions, algebraic manipulations, and approximation formulae.
6. Investigate the establishment of mathematical research centres, particularly astronomical observatories, during the era of Jainism, led by scholars like Aryabhata and Varahamihira.
7. Discuss Brahmagupta's contributions, including introducing zero as a number, developing interpolation techniques, and presenting formulas for solving equations.
8. Explore the 12th-century contributions of Bhaskara II in trigonometry, number theory, algebra, and astronomy, as well as the subsequent development of these ideas by other Indian scholars.
9. Examine the 14th-century scholar Madhava's significant contributions to analysis, including infinite series expansions, the binomial theorem, and approximations for π.
10. Trace the spread of the decimal system beyond India, its mention in a Syrian manuscript in 662, and its adoption in Cambodia and other Asian countries before becoming common in the Islamic world and eventually reaching Europe. Discuss the role of Arab scholars in preserving and translating Indian mathematical texts.

Activity 191. Write an overview of the text in Activity 186 using Appendix II.

Unit 25. Arabic Mathematics

Activity 192. Do the quiz on Arabic mathematics. In pairs, compare your answers.

1. **Who established the House of Wisdom in Baghdad in the 8th century?**
   a. Archimedes
   b. Caliph al-Ma'mun
   c. Euclid
   d. Muhammad ibn Musa al-Khwarizmi
2. **What was the House of Wisdom known for?**
   a. destruction of mathematical texts
   b. repository of important academic texts
   c. promoting Greek philosophy
   d. developing modern algebra
3. **Which Greek mathematician's work had a tremendous impact on Arab scholars at the House of Wisdom?**
   a. Archimedes
   b. Diophantus
   c. Euclid
   d. Menelaus of Alexandria
4. **What did al-Khwarizmi's work "Calculation by Restoration and Reduction" contribute to mathematics?**
   a. number theory
   b. algebra
   c. geometry
   d. trigonometry
5. **What departure did al-Khwarizmi make from Greek thinking?**
   a. introduction of geometry
   b. unification of arithmetic and geometry through algebra
   c. focus on trigonometry
   d. disregard for numerical representation
6. **Who refined approaches for reducing geometric problems to algebraic ones?**
   a. al-Khwarizmi
   b. al-Ma'mun
   c. al-Mahani
   d. Omar Khayyam

7. **What did Omar Khayyam attempt to develop methods for?**
    a. solving quadratic equations
    b. solving cubic equations
    c. calculating trigonometric tables
    d. generating amicable numbers
8. **Which mathematician attempted to classify all even perfect numbers?**
    a. al-Farisi
    b. al-Haytham
    c. Ibrahim ibn Sinan
    d. Thabit ibn Qurra
9. **What did Thabit ibn Qurra find a method for generating?**
    a. Fibonacci sequence
    b. prime numbers
    c. amicable numbers
    d. Pascal's triangle
10. **What did scholars in the Islamic Golden Age contribute to numeric computations?**
    a. advancements in calculus
    b. effective methods for decimal place-value representation
    c. development of the quadratic formula
    d. exploration of non-Euclidean geometries
11. **Which scholar developed effective methods for extracting the n-th root of a number?**
    a. al-Farisi
    b. al-Haytham
    c. Jamshid al-Kashi
    d. Omar Khayyam
12. **What did Ibrahim ibn Sinan introduce a method of?**
    a. differentiation
    b. integration
    c. prime factorization
    d. trigonometry

Activity 193. Read the article. Review your answers to the quiz in Activity 192.

Mathematical historians of today are grateful to the Arabic scholars of the past for preserving, translating, and honouring the great Indian, Greek, and Islamic mathematical works of the scholars before them, and for their own significant contributions to the development of mathematics. At the end of the 8th century, with the great Library of Alexandria destroyed, Caliph al-Ma'mun set up a House of Wisdom in Baghdad, Iraq, which became the next prominent centre of learning and research, as well as the repository of important academic texts. Many scholars were employed by the caliph to translate the mathematical works of the past and develop further the ideas they contained. As the Islamic empire grew over the following seven centuries, the culture of intellectual pursuit also spread. Many scholars of 12th-century Europe, and later, visited the Islamic libraries of Spain to read the texts of the Arabic academics and to learn of the advances that had occurred in the East during the dark ages of the West. A significant amount of mathematical material was transmitted to Europe via these means.

One of the first Greek texts to be translated at the House of Wisdom was Euclid's famous treatise, "The Elements". This work made a tremendous impact on the Arab scholars of the period, and many of them, when conducting their own research, formulated theorems, and proved results precisely in the style of Euclid. Members of the House of Wisdom also translated the works of Archimedes of Syracuse, Diophantus of Alexandria, Menelaus of Alexandria, and others, and so they were certainly familiar with all the great Greek advances in the topics of geometry, number theory, mechanics, and analysis. They also translated the works of Indian scholars, Aryabhata and Bhaskara, for instance, and were familiar with the theory of trigonometry, methods in astronomy, and further topics in geometry and number theory. Any Arab scholar who visited the House of Wisdom had, essentially, the entire bulk of human mathematical knowledge available to him in his own language.

The Arab mathematician Muhammad ibn Musa al-Khwarizmi (ca. 800) wrote a number of original texts that were enormously influential. His first piece simply described the decimal place-value system he had learned from Indian sources. Three hundred years later, when translated into Latin, this work became the primary source for Europeans who wanted to learn the new system for writing and manipulating numbers. But more important was al-Khwarizmi's piece "Calculation by Restoration and Reduction", from which the topic of algebra arose. Al-Khwarizmi was fortunate to have all sources of mathematical knowledge available to him. He began to see that the then-disparate notions of "number" and "geometric magnitude" could be unified as one whole by developing the concept of algebraic objects. This represented a significant departure from Greek thinking, in which mathematics is synonymous with geometry. Al-Khwarizmi's insight provided a means to study both arithmetic and geometry under a single framework, and his methods of algebra paved the way for significant developments in mathematics of much broader scope than ever previously envisioned.

The mathematician al-Mahani (ca. 820) developed refined approaches for reducing geometric problems to algebraic ones. He showed, in particular, that the famous problem of duplicating the cube is essentially an algebraic issue. Other scholars brought rigour to the subject by proving that certain popular, but complicated, algebraic methods were valid. These scholars were comfortable manipulating polynomials and developed rules for working with exponents. They solved linear and quadratic equations, as well as various systems of equations. Surprisingly, no one of the time thought to ease matters by using symbols to represent quantities: all equations and all manipulations were described fully in words each and every time they were employed.

With quadratic equations well understood, the scholar Omar Khayyam (ca. 1048–1131) attempted to develop methods of solving degree-three equations. Although he was unable to develop general algebraic methods for this task, he did find ingenious geometric techniques for solving certain types of cubics with the aid of conic sections. He was aware that such equations could have more than one solution.

In number theory, Thabit ibn Qurra (ca. 836–901) found a beautiful method for generating amicable numbers. This technique was later utilized by al-Farisi (ca. 1260–1320) to yield the pair 17,296 and 18,416, which today is usually attributed to Leonhard Euler (1707–83). In his writing, Omar Khayyam referred to earlier Arab texts, now lost, that discuss the equivalent of Pascal's triangle and its connections to the binomial theorem. The mathematician al-Haytham (ca. 965–1040) attempted to classify all even perfect numbers.

Taking advantage of the ease of the Indian system of decimal place-value representation, Arabic scholars also made great advances in numeric computations. The great 14^{th}-century scholar Jamshid al-Kashi developed effective methods for extracting the n-th root of a number and evaluated π to a significant number of decimal places. Scholars at the time also developed effective methods for computing trigonometric tables and techniques for making highly accurate computations for the purposes of astronomy.

On a theoretical note, scholars also advanced the general understanding of trigonometry and explored problems in spherical geometry. They also investigated the philosophical underpinnings of geometry, focusing, in particular, on the role the famous parallel postulate plays in the theory. Omar Khayyam, for instance, attempted to prove the parallel postulate — failing, of course — but did accidentally prove results about figures in non-Euclidean geometries along the way. The mathematician Ibrahim ibn Sinan (908–946) also introduced a method of "integration" for calculating volumes and areas following an approach more general than that developed by Archimedes of Syracuse (ca. 287–212 B.C.E.). He also applied his approach to the study of conic sections and to optics.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 194. Identify the individuals based on the descriptions from the text in Activity 193.

al-Farisi / al-Haytham / al-Mahani / Archimedes of Syracuse / Aryabhata / Bhaskara / Diophantus of Alexandria / Ibrahim ibn Sinan / Jamshid al-Kashi / Menelaus of Alexandria / Muhammad ibn Musa al-Khwarizmi / Omar Khayyam / Thabit ibn Qurra

1. An Arab mathematician who introduced a method of "integration" for calculating volumes and areas, with applications to conic sections and optics.
2. Arab mathematician who utilized Thabit ibn Qurra's method to find an amicable number pair.
3. Arab mathematician who attempted to classify even perfect numbers and made contributions to various mathematical fields.
4. Arab mathematician of the 14th century who developed effective methods for extracting roots and calculated π to many decimal places.
5. Arab mathematician who attempted to develop methods for solving cubic equations and made contributions to number theory.
6. Arab mathematician who developed refined approaches for reducing geometric problems to algebraic ones.
7. Arab mathematician who found a method for generating amicable numbers in number theory.
8. Arab mathematician who wrote influential texts on the decimal place-value system and algebra.
9. Greek mathematicians whose works were translated at the House of Wisdom, contributing to Arab scholars' knowledge.
10. Indian scholars whose works were translated at the House of Wisdom, introducing trigonometry and astronomy to Arab scholars.

Activity 195. Rearrange the events in chronological order according to the text of Activity 193. Provide dates where possible.

a. Al-Mahani develops refined approaches for reducing geometric problems to algebraic ones, highlighting the algebraic nature of the famous problem of duplicating the cube.

b. Ibrahim ibn Sinan introduces a method of "integration" for calculating volumes and areas.
c. Islamic libraries in Spain attract European scholars.
d. Jamshid al-Kashi advances numeric computations, extracting the n-th root of a number and calculating π to a significant number of decimal places.
e. Muhammad ibn Musa al-Khwarizmi writes influential texts, describing the decimal place-value system and introducing algebra through "Calculation by Restoration and Reduction."
f. Omar Khayyam attempts to develop methods for solving degree-three equations and introduces ingenious geometric techniques for certain cubics using conic sections.
g. Omar Khayyam attempts to prove the parallel postulate, unintentionally contributing to non-Euclidean geometries.
h. Thabit ibn Qurra discovers a method for generating amicable numbers.

Activity 196. Determine whether the statements are true or false by quoting from the text in Activity 193.

1. "The Elements" by Euclid was among the initial Greek texts translated at the House of Wisdom, leaving a notable impact on Arab scholars.
2. Al-Khwarizmi's contribution to the decimal place-value system became a key source for Europeans when translated into Latin.
3. Al-Khwarizmi's insight significantly contributed to the unification of algebra and geometry under a single framework.
4. Al-Mahani developed and improved methods for converting algebraic problems into geometric forms.
5. Arabic scholars described equations and manipulations using algebraic notation.
6. Contemporary mathematical historians acknowledge the significance of ancient Arabic scholars in preserving and translating mathematical works from diverse traditions.
7. Following the destruction of the Library of Alexandria, Caliph al-Ma’mun founded the House of Wisdom in Baghdad, which emerged as a significant centre for learning.
8. Omar Khayyam was unsuccessful in developing general algebraic methods for solving degree-three equations.
9. Thabit ibn Qurra's method for generating amicable numbers had a notable influence on later mathematicians.

Activity 197. In groups, discuss the points. Refer to the text in Activity 193.

1. Discuss the role of the House of Wisdom in Baghdad, Iraq, and its significance as a centre for preserving, translating, and developing mathematical knowledge from

Indian, Greek, and Islamic traditions. How did the destruction of the Library of Alexandria contribute to the establishment of such centres?

2. Explore the impact of translating Greek texts, including Euclid's "The Elements," at the House of Wisdom. How did this influence Arab scholars and contribute to the development of geometry, number theory, mechanics, and analysis in the Islamic world?
3. Examine the contributions of Muhammad ibn Musa al-Khwarizmi to the field of mathematics, particularly in the context of his work on the decimal place-value system and the emergence of algebra. How did al-Khwarizmi's insights represent a departure from Greek mathematical thinking?
4. Analyze the development of algebraic methods by al-Mahani and other scholars, including their approaches to reducing geometric problems to algebraic ones. Discuss the significance of their contributions to solving equations and manipulating polynomials.
5. Explore Omar Khayyam's attempt to develop methods for solving degree-three equations and his ingenious geometric techniques for certain types of cubics using conic sections. How did Khayyam's work contribute to the understanding of cubic equations?
6. Discuss the method for generating amicable numbers found by Thabit ibn Qurra and later utilized by al-Farisi. How did this technique contribute to the study of number theory, and what are the connections to later mathematicians like Leonhard Euler?
7. Explore the advancements made by Arabic scholars in numeric computations, including the effective methods for extracting the n-th root of a number by Jamshid al-Kashi. How did these developments contribute to practical applications and mathematical accuracy?
8. Discuss the theoretical advancements in trigonometry and spherical geometry by Islamic scholars. How did their exploration of the parallel postulate and attempts to prove it contribute to the understanding of non-Euclidean geometries?
9. Explore Ibrahim ibn Sinan's method of "integration" for calculating volumes and areas, and its generality compared to Archimedes' approach. How did this contribute to the theoretical understanding of mathematical concepts beyond what was known in earlier times?
10. Discuss the interdisciplinary nature of mathematical exploration in the Islamic world, incorporating optics and other fields. How did this interdisciplinary approach contribute to the overall development of mathematics and its applications in various domains?

Activity 198. Write an overview of the text in Activity 193 using Appendix II.

“Nature is written in mathematical language.” (Galileo Galilei)

Module 6. Mathematics and Related Fields

Unit 26. Physics and Physical Science

Activity 199. In pairs, discuss the questions.

1. Does a mathematician have to excel at physics?
2. Does a physicist have to excel at mathematics?
3. What is the difference between physics and physical science?
4. How are physics and mathematics related?

Activity 200. Match the words with the definitions.

1. matter
2. motion
3. energy

a. the capacity of a body or system to do work
b. the physical part of the universe consisting of solids, liquids, and gases
c. the process of continual change in the physical position of an object

Activity 201. Complete the tables.

Table 32. Base SI Units

amount of a substance / angle / electric current / length / light intensity / mass / solid angle / temperature / time		
Base Units	**Name**	**Symbol**
(1) ___________	meter	m
(2) ___________	kilogram	kg

(3) ___________	second	s
(4) ___________	Kelvin	K
(5) ___________	mole	mol
(6) ___________	ampere	A
(7) ___________	candela	cd
(8) ___________	radian	rad
(9) ___________	steradian	sr

Table 33. Conversion Between Metric and Imperial Systems

1,000 / 1,609 / 30 / 2.54 / 0.91 / 236.588 / 568.261 / 453.592 / 946.353 / 28.3495 / 29.5735 / 3.78541	
Metric System	**Imperial System**
Distance	
(10) _____ cm	1 in (inch) / 1”
(11) _____ cm	1 ft (foot) / 1’ = 12 in
(12) _____ m	1 yd (yard) = 3 ft
(13) _____ km	1 mi (mile) = 1,760 yd
Mass	
(14) _____ g	1 oz (ounce)
(15) _____ g	1 lb (pound) = 16 oz
(16) _____ kg	1 ton = 2204,62 lb
Volume	
(17) _____ ml	1 fl oz (fluid ounce)
(18) _____ ml	1 cup = 8 fl oz
(19) _____ ml	1 pt (pint) = 2.4019 cup
(20) _____ ml	1 qt (quart) = 2 pt
(21) _____ l	1 gal (gallon) = 4 qt

Activity 202. Read the article. Expand on the physical and abstract nature of mathematics as a science.

Physics is a science that deals with the structure of matter and the interactions between the fundamental constituents of the observable universe. In the broadest sense, physics is concerned with all aspects of nature on both the macroscopic and submicroscopic levels. Its scope of study encompasses not only the behaviour of objects under the action of given forces but also the nature and origin of gravitational, electromagnetic, and nuclear force fields. Its ultimate objective is the formulation of a few comprehensive principles that bring together and explain all such disparate phenomena.

Physics is the basic physical science. Until rather recent times physics and natural philosophy were used interchangeably for the science whose aim is the discovery and formulation of the fundamental laws of nature. As the modern sciences developed and became increasingly specialized, physics came to denote that part of physical science not included in astronomy, chemistry, geology, and engineering. Physics plays an important role in all the natural sciences, however, and all such fields have branches in which physical laws and measurements receive special emphasis, bearing such names as astrophysics, geophysics, biophysics, and even psychophysics.

Physics can, at base, be defined as the science of matter, motion, and energy. Its laws are typically expressed with economy and precision in the language of mathematics. Although mathematics is used throughout the physical sciences, it is often debated whether mathematics is itself a physical science. Those who include it as a physical science point out that physical laws can be expressed in mathematical terms and that the concept of number arises in counting physical objects. Those who say mathematics is not a physical science consider numbers as abstract concepts that are helpful in describing groups of objects but do not arise from the physical objects themselves.

The ultimate aim of physics is to find a unified set of laws governing matter, motion, and energy at small (microscopic) subatomic distances, at the human (macroscopic) scale of everyday life, and out to the largest distances (e.g., those on the extragalactic scale). This ambitious goal has been realized to a notable extent. Although a completely unified theory of physical phenomena has not yet been achieved (and possibly never will be), a remarkably small set of fundamental physical laws appears able to account for all known phenomena. The body of physics developed up to about the turn of the 20th century, known as classical physics, can largely account for the motions of macroscopic objects that move slowly with respect to the speed of light and for such phenomena as heat, sound, electricity, magnetism, and light. The modern developments of relativity and quantum mechanics modify these laws insofar as they apply to higher speeds, very massive objects, and to the tiny elementary constituents of matter, such as electrons, protons, and neutrons.

(from Encyclopaedia Britannica)

Table 34. Multiples of Units

Multiples	Prefix	Symbol	Origin
10^{24}	yotta-	Y	Greek “eight times” (24 = 8 × 3)
10^{21}	zetta-	Z	Greek “seven times” (21 = 7 × 3)
10^{18}	exa-	E	Greek “six times” (18 = 6 × 3)
10^{15}	peta-	P	Greek “five times” (15 = 5 × 3)
10^{12}	tera-	T	Greek “monster”
10^{9}	giga-	G	Greek “giant”
10^{6}	mega-	M	Greek “big”
10^{3}	kilo-	k	Greek “thousand”
10^{2}	hecto-	h	Greek “hundred”
10	deka-	da	Greek “ten”
10^{-1}	deci-	d	Latin “ten”
10^{-2}	centi-	c	Latin “hundred”
10^{-3}	milli-	m	Latin “thousand”
10^{-6}	micro-	μ	Greek “small”
10^{-9}	nano-	n	Greek “dwarf”
10^{-12}	pico-	p	Italian “small”
10^{-15}	femto-	f	Danish “15”
10^{-18}	atto-	a	Danish “18”
10^{-21}	zepto-	z	Greek “seven times”
10^{-24}	yocto-	y	Greek “eight times”

Activity 203. Finish the sentences from the text in Activity 202.

1. Physics is a science that deals with…
2. In the broadest sense, physics is concerned with…
3. Its scope of study encompasses…
4. Its ultimate objective is…
5. Physics plays an important role in…
6. Physics can, at base, be defined as…
7. Its laws are typically expressed…
8. The ultimate aim of physics is…

Activity 204. Do you believe that mathematics was discovered or created? Watch the video "Is Math Discovered or Invented?" to choose the best answer to the questions. Then watch the video again and make a note of all the proponents advocating for each point of view.

https://disk.yandex.ru/i/9_dzLe74Bp20pg

1. What did the Pythagoreans believe about numbers?
A. Numbers were invented by humans to measure things
B. Numbers were living entities and universal principles
C. Numbers only existed in mathematical equations
D. Numbers were useful but had no real meaning

2. What was Leopold Kronecker's view on mathematics?
A. All mathematical concepts were created by God
B. Mathematics exists independently in nature
C. Only natural numbers were divine; everything else was created by humans
D. Mathematics was discovered through scientific experiments

3. What does the phrase "the unreasonable effectiveness of mathematics" suggest?
A. Mathematics is too difficult for most people to understand
B. Mathematical theories often describe physical phenomena they weren't designed to explain
C. Mathematics should not be used in physics
D. Mathematical rules are unreasonable and need to be changed

4. Why is Godfrey Hardy's work mentioned?
A. To show that theoretical mathematics can later prove useful in real-world applications
B. To demonstrate that mathematics is always practical
C. To prove that number theory has no value
D. To explain why he won a Nobel Prize in mathematics

5. What is the main purpose of the video?
A. To prove that mathematics was definitely invented by humans
B. To teach readers how to solve mathematical problems
C. To present different perspectives on whether mathematics is discovered or invented
D. To explain why ancient mathematicians were wrong about numbers

Activity 205. In writing, comment on the citation.

“The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.”

(by Eugene Wigner, from “The Unreasonable Effectiveness of Mathematics in the Natural Sciences,” 1960)

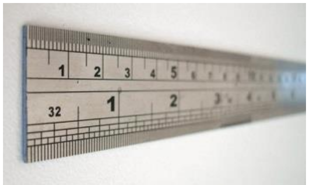
Figure 10. Eugene Wigner

Unit 27. Computer Science

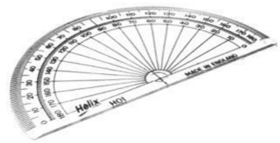
Activity 206. Match the words with the images.

1) a compass
   (a pair of compasses)
2) a protractor
3) a ruler
4) a set square (BrE)
   a triangle (AmE)

a)



b)



c)

d)

Activity 207. Match the words with the definitions.

1. abacus
2. calculator
3. computer

a. a device that is able to store a number, add it to another number, and mechanically produce the result, taking care of any carried digits
b. an electronic device for automatically performing either arithmetic operations on data or sequences of manipulations on sets of symbols (as required for algebra and set theory, for instance), all according to a precise set of predetermined instructions
c. any counting board with beads laid in parallel grooves or strung on parallel rods

Activity 208. In pairs, discuss the questions.

1. Is the computer an integral instrument of a mathematician? Why?
2. What tools and devices do mathematicians use? Why?
3. What computer software and smartphone applications do mathematicians use? Why?
4. How are computer science and mathematics related?

Activity 209. Read the article to expand on the interdisciplinary connection and mutual influence that exists between mathematics and computer science.

The roots of modern computer science lie in an interest in rapid computation. Simple mechanical calculators may date back to ancient times; however, it is the work of mathematicians Blaise Pascal (1623–1662) and Gottfried Leibniz (1646–1716) that gave rise to the first practical mechanical calculators. By the mid-19th century, Charles Babbage (1791–1871) had conceptualized and designed mechanical computers that included the essential features (programs, processor, memory, input/output) of the modern digital computer. His motivation was the need for rapid, accurate calculation of statistical tables made necessary by the manufacturing economy of the Industrial Revolution. By the end of the century, the volume of such data had increased to the point where mechanical calculators and tabulators had become the only practical way to keep up.

Mathematically, a computer can be seen as a way to rapidly and automatically execute procedures that have been proven to lead to reliable solutions to a problem. Once computers came on the scene, mathematical principles for verifying or proving algorithms would acquire new practical importance.

By the early 20th century, however, mathematicians were beginning to examine the problem of determining what propositions were provable, and in 1931 Kurt Gödel published a proof that any mathematical system necessarily allowed for the formation of propositions that could not be proven using the axioms of that system. An analogous question was determining what problems were computable. Working independently, two researchers formulated models that could be used to test for computability. Turing's model, in particular, provided a theoretical construct that could, using combinations of a few simple operations, calculate anything that was computable.

By the 1940s, electromechanical (relays) or electronic (tube) switching elements made it possible to build practical high-speed computers. Computer circuit designers could draw upon the advances in symbolic logic in the 19th century. Boolean logic, with its true/false values, would prove ideal for operating computers constructed from on/off switched elements.

The mathematical tools of the previous 150 years could now be used to design systems that could not only calculate but also manipulate symbols and achieve results in higher mathematics.

A variety of mathematical disciplines bear upon the design and use of modern computers. Simple or complex algebra using variables in formulas is at the heart of many programs ranging from financial software to flight simulators.

Geometry, particularly the analytical geometry based upon the coordinate system devised by René Descartes (1596–1650) is fundamental to computer graphics displays, where the screen is divided into X (vertical) and Y (horizontal) axes. Modern graphics systems have added 3D depiction and sophisticated algorithms to allow the rapid display of complex objects. Beyond graphics, the Cartesian insight that converted geometry into algebra makes a variety of geometrical problems accessible to computation, including the finding of optimum paths for circuit design. Design of computer and network architectures also involves the related field of topology. The fascinating field of fractal geometry has found use in computer graphics and data storage techniques.

Aspects of number theory, often considered the most abstract branch of mathematics, have found surprising relevance in computer applications. These include randomization (random number generation) and the factoring of large numbers, which is crucial for cryptography.

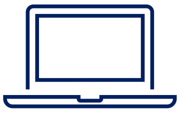
Mathematics as a discipline is thus essential to its younger sibling, computer science. In turn, however, computer science and technology have enriched the pursuit of mathematical truth in surprising ways. As early as 1956, a program called Logic Theorist, written by Herbert Simon (1916–2001) and Allen Newell (1927–1992) demonstrated how a program (that is, a collection of algorithms) could prove mathematical propositions given axioms and rules. While these early programs worked on a somewhat hit-or-miss basis, later theorem-solving programs produced solutions different from the standard ones known to mathematicians, and sometimes more elegant. Thus, the computer, which began as an aid to calculation, became an aid to symbol manipulation and to some extent an independent creative source.

(by Harry Henderson, from Encyclopedia of Computer Science and Technology)

Activity 210. Reorder the sentences to make a text on cryptography.

a. If letters of the alphabet and punctuation marks are replaced by numbers, then mathematics can be used to create effective codes.

b. In 1977 three mathematicians, Ron Rivest, Adi Shamir, and Leonard Adleman, developed a public-key cryptography method in which the method of encoding a message can be public to all without compromising the security of the message.
c. It is the primary encryption method used today by financial institutions to transmit sensitive information across the globe.
d. On the other hand, multiplying and raising large numbers to powers is easy for computers to do, and so the RSA method is also very easy to implement.
e. The practice of altering the form of a message by codes and ciphers to conceal its meaning to those who intercept it, but not to those who receive it, is called cryptography.
f. The RSA encryption method, as it is known today, is based on the mathematics of the modular arithmetic and relies on the fact that it is extraordinarily difficult to find the two factors that produce a given large product.
g. The RSA system is extremely secure.



Activity 211. What is natural language processing, speech recognition, and speech synthesis? Watch the video "The Turing Test: Can a Computer Pass For a Human?" to choose the best answer to the questions. Describe the purpose, procedure, and participants of the test.

https://disk.yandex.ru/i/vZNyknipXUWCUQ

1. What was Alan Turing's main approach to measuring artificial intelligence?
A. He focused on whether machines could develop consciousness
B. He tested whether computers could communicate like humans
C. He measured the memory capacity of different computers
D. He studied the neurons in artificial brains

2. Why did ELIZA successfully fool many people during conversations?
A. It had more memory than other programs at that time
B. It used complex mathematical equations to generate responses
C. It acted like a psychologist and reflected questions back to users
D. It could discuss any topic with detailed knowledge

3. What weakness of the Turing Test do ELIZA and PERI demonstrate?
A. Computers need too much memory to pass the test
B. People often think things are intelligent when they really aren't
C. Judges always know when they're talking to a machine
D. Programs can't have conversations about multiple topics

4. Why does human conversation remain difficult for computers?
A. Modern computers don't have enough processing power yet
B. Language involves complex understanding that goes beyond dictionary definitions
C. Programmers haven't created large enough databases of conversations
D. Computers can't remember previous conversations with users

5. What can be inferred about Turing's prediction for the year 2000?
A. He accurately estimated the memory computers would need
B. He underestimated how difficult human conversation would be for machines
C. He knew that computers would focus on fooling judges
D. He expected machines to develop consciousness by that time

Activity 212. Choose one technological innovation applicable to mathematical research and education. Present its functionality with multimedia slides.

Unit 28. Economics

Activity 213. Choose the best alternative.

1. A small car is more economic / economical to run.
2. Buy the large economy / economical pack.
3. Economic / Economical growth is slow.
4. He bought the economy / economical size and saved money.
5. He is studying economics / economy.
6. He thought I had been economic / economical with the truth.
7. Tourism is an important part of the economics / economy.

Activity 214. Complete the passage with the words in the box.

distribute / goods and services / market / produce / rate / resources / scarce / supply and demand

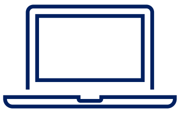
In the realm of mathematics, the principles of (1) ___________ find a unique application. Much like in a (2) ___________ where the availability of (3) ___________ must meet consumer demand, mathematical problems involve a balance of resources and requirements. Equations and formulas act as the tools to (4) ___________ solutions, with the (5) ___________ at which these solutions are obtained reflecting the efficiency of mathematical processes. (6) ___________, such as time and computational power, are often (7) ___________, prompting mathematicians to (8) ___________ their efforts wisely.

Activity 215. Read the passage. In pairs, discuss the questions in the box.

Economics is a social science that analyzes and describes the consequences of choices made concerning scarce productive resources. Economics is the study of how individuals and societies choose to employ those resources: what goods and services will be produced, how they will be produced, and how they will be distributed among the members of society. Economics is customarily divided into microeconomics and macroeconomics. Of major concern to macroeconomists are the rate of economic growth, the inflation rate, and the rate of unemployment. Specialized areas of economic investigation attempt to answer questions on a variety of economic activity; they include agricultural economics, economic development, economic history, environmental economics, industrial organization, international trade, labour economics, money supply and banking, public finance, urban economics, and welfare economics. Specialists in mathematical economics and econometrics provide tools used by all economists. The areas of investigation in economics overlap with many other disciplines, notably history, mathematics, political science, and sociology.

(from Encyclopaedia Britannica)

1. What is economics?
2. What branches does economics comprise?
3. What role does mathematics play in economics?



Activity 216. Read the article to match the headings (a–c) with the sections (1–3).

(a) Applications of mathematics in economics

(b) Branches of mathematics integral to economics

(c) Role of mathematics in economics

(1) ________________________________

Although mathematics is indispensable to all types of economics, it's most common in mathematical economics, where it's a core component. In mathematical economics, economists apply mathematical principles to economic theory. An economist may use mathematics alongside other methods and tools and techniques, such as data harvesting and computer algorithms.

Mathematics in economics allows an economist to offer more precision with their projections and analysis. This may allow them to extract increased guidance from the results of their analysis. Using hard data and mathematical calculations can also reduce the potential for bias in economic projections.

The importance of mathematics in economics has increased with the growing presence of computing in the field. Computer technology allows economists to process large amounts of data or more complex mathematical equations more easily. This expands the capability of mathematics in economics as computers can make complex calculations easier to complete.

(2) ______________________________

Algebra is a basic math field, and it serves as a foundation for many other forms of mathematical calculation. Algebra allows an individual to solve equations with one or more variables, finding the result for a variable under defined conditions. For an economist, algebra is a useful mathematical skill for calculation and projection. Working with variables allows an economist to perform a task such as setting a target growth rate and solving for the required related variables to reach that rate using an established equation.

Calculus is a mathematical field dealing with rate-of-change calculations. Calculus can be a powerful tool for an economist when assessing economic performance and making projections. Using calculus to generate curves based on economic information allows one to identify trends and make more informed decisions. As an economist, one may apply this to projects such as market assessment, supply and demand analysis and economic forecasting.

The mathematical field of probability measures the likelihood of a specific occurrence or outcome. Probabilistic assessment allows a mathematician to identify whether a potential outcome is likely to occur and to compare the relative likelihood of two or more potential outcomes. As an economist, understanding probability can be useful when making estimations of likely outcomes to economic decisions. By combining the probability of different potential outcomes multiplied by the estimated cost or benefit of each outcome, one can perform an assessment of the expected cost or benefit of an economic plan.

Statistics is a mathematical study that focuses on the collection, sorting and analysis of sets of data. It allows a mathematician to assess a population represented within the data. That is a critical skill for tasks such as modelling and projecting for behaviours or responses within a community. Statistics is a valuable mathematical skill for an economist because it allows one to work with large amounts of data. Applying statistical calculations to datasets may allow an economist to identify key trends or information on grouping within a community. One may then use this information to guide oneself or others in decision-making positions when making suggestions for plans or policies.

(3) ______________________________

Analysis is a key responsibility for an economics professional. Economic work often includes assessing information about economic performance, markets and other key economic data and extrapolating relevant information from the data. This allows individuals making economic decisions to do so while using information from various sources. Math plays a large part in many forms of data analysis. This can include both the simple mathematics to perform tasks such as finding averages to advanced mathematics in the form of differential equations. Strong math skills in a diverse range of math capabilities can help an economist complete their analytical work more effectively.

An economic model provides a visualization of key economic data. Using a model can make it easier for individuals to conceptualize or understand the state of an economic market. A model may also provide a new form for data that offers insights that the raw dataset does not. An economist is likely to use their math skills throughout the process of creating an economic model. Accurate math provides reliable data one can use in constructing a model, which can increase the value of the model provides upon completion.

Economic projections provide predictions of future economic behaviour and patterns. Accurate projections are a valuable tool for economists, as it allows them to make decisions for future planning based on the state and behaviour of the market in the future. Math is an integral part of creating economic projections. It allows an economist to perform calculations on economic data, often using the principles of calculus to assess potential changes in the data over time. Developing mathematical skills as an economist can help improve the accuracy of calculations both by ensuring one completes them correctly and expanding the number of calculations and math principles one understands and may apply to one's work.

(from indeed.com)

Activity 217. Outline mathematics in economics by developing the points.

1. mathematical economics
2. computers
3. algebra
4. calculus
5. probability
6. statistics
7. analyzing
8. modelling
9. projecting

Activity 218. Reorder the sentences to make a text on econometrics.

a. Econometricians estimate production functions and cost functions for firms, supply-and-demand functions for industries, income distribution in an economy, macroeconomic models and models of the monetary sector for policy makers, and business cycles and growth for forecasting.
b. Econometrics creates equations to describe phenomena such as the relationship between changes in price and demand.
c. Econometrics is the statistical and mathematical analysis of economic relationships, often serving as a basis for economic forecasting.
d. Information derived from these models helps both private businesses and governments make decisions and set monetary and fiscal policy.
e. It is used mainly, however, by economists to study relationships between economic variables.
f. Such information is sometimes used by governments to set economic policy and by private business to aid decisions on prices, inventory, and production.

Activity 219. Draw a mind map of the text in Activity 216. You can use the following resources:

https://app.diagrams.net/

https://freemind.sourceforge.net/wiki/index.php/Main_Page

https://simplemind.eu/

https://coggle.it/

https://www.mindmup.com/

https://miro.com/ru/

Unit 29. Mathematics in Non-Fiction

Activity 220. In pairs, discuss the questions.

1. Why did you decide to specialize in mathematics?
2. Did you dream of becoming a mathematician?
3. Did you like your teacher(s) of mathematics? Why?
4. How has your attitude to mathematics changed over the years?

Activity 221. Match the influential mathematical books with their authors. Characterize the contents and impact of one text.

1. A Course of Pure Mathematics
2. Arithmetica
3. Ars Magna
4. Elements of Mathematics
5. La Géométrie
6. Liber Abaci
7. Principia Mathematica
8. Statistical Methods for Research Workers
9. Synagoge / Collection
10. The Almagest
11. The Compendious Book on Calculation by Completion and Balancing / Al-Jabr
12. The Doctrine of Chances
13. The Elements
14. The Ground of Arts
15. The Laws of Thought
16. Theory of Games and Economic Behavior

a. Abraham de Moivre
b. Alfred North Whitehead and Bertrand Russell
c. Claudius Ptolemy
d. Diophantus of Alexandria
e. Euclid
f. Fibonacci
g. G. H. Hardy
h. George Boole
i. Gerolamo Cardano
j. John von Neumann and Oskar Morgenstern
k. Muhammad ibn Musa al-Khwarizmi
l. Nicolas Bourbaki
m. Pappus of Alexandria
n. René Descartes
o. Robert Recorde
p. Sir Ronald Aylmer Fisher

Activity 222. What is a polymath? Watch the video "The Greatest Mathematician That Never Lived" to choose the best answer to the questions. Write a summary of Nicolas Bourbaki's identity and mathematical legacy.

https://disk.yandex.ru/i/R3HeCOX51AvpoA

1. Why was Nicolas Bourbaki's application to the American Mathematical Society rejected?
A. His published work was not considered important enough
B. He was not a real person
C. He refused to meet with the committee members
D. His textbooks contained too many errors

2. What was the main problem with mathematics before the group formed?
A. There were too many mathematicians working in the field
B. Mathematical textbooks were too expensive for students
C. Different branches of mathematics lacked a common framework
D. Universities refused to teach advanced mathematical concepts

3. What does the is suggested about the group's approach to mathematics?
A. They believed mathematics should be based on intuition rather than logic
B. They thought that strict logical systems would limit mathematical creativity
C. They challenged the common view that mathematics was primarily intuitive
D. They wanted to separate mathematics into more specialized branches

4. What is a bijective function?
A. A function where multiple inputs can produce the same output
B. A function where each element has a one-to-one correspondence
C. A function where outputs cannot be mapped back to inputs
D. A function that only works with numerical values

5. How did the group maintain the illusion that Bourbaki was real?
A. They hired an actor to appear at mathematical conferences
B. They created elaborate stories and sent communications in his name
C. They published false biographical information in academic journals
D. They claimed he had won several international mathematics awards

Activity 223. Read the excerpt from the 1940 essay "A Mathematician's Apology" by British mathematician G. H. Hardy. Compare your mathematical journey to that of G. H. Hardy. Are there ideas in the text that resonate with you? Identify any similarities and/or differences between you and the author. In pairs, exchange your thoughts.

Figure 11. G. H. Hardy

I cannot remember ever having wanted to be anything but a mathematician. I suppose that it was always clear that my specific abilities lay that way, and it never occurred to me to question the verdict of my elders. I do not remember having felt, as a boy, any passion for mathematics, and such notions as I may have had of the career of a mathematician were far from noble. I thought of mathematics in terms of examinations and scholarships: I wanted to beat other boys, and this seemed to be the way in which I could do so most decisively.

I found at once, when I came to Cambridge, that a Fellowship implied "original work", but it was a long time before I formed any definite idea of research. I had of course found at school, as every future mathematician does, that I could often do things much better than my teachers; and even at Cambridge, I found, though naturally much less frequently, that I could sometimes do things better than the College lecturers. But I was really quite ignorant, even when I took the Tripos of the subjects on which I have spent the rest of my life; and I still thought of mathematics as essentially a "competitive" subject. My eyes were first opened by Professor Love, who taught me for a few terms and gave me my first serious conception of analysis. But the great debt which I owe to him was his advice to read Jordan's famous "Cours d'Analyse"; and I shall never forget the astonishment with which I read that remarkable work, the first inspiration for so many mathematicians of my generation, and learnt for the first time as I read it what mathematics really meant. From that time onwards, I was in my way a real mathematician, with sound mathematical ambitions and a genuine passion for mathematics.

I wrote a great deal during the next ten years, but very little of any importance; there are not more than four or five papers which I can still remember with some satisfaction. The real crisis of my career came ten or twelve years later, in 1911, when I began my long collaboration with Littlewood, and in 1913, when I discovered Ramanujan. All my best work since then has been bound up with theirs, and it is obvious that my association with them was the decisive event of my life. I still say to myself when I am depressed and find myself forced to listen to pompous and tiresome people, "Well, I have done one the thing you could never have done, and that is to have collaborated with both Littlewood and Ramanujan on something like equal terms." It is to them that I owe an unusually late maturity: I was at my best a little past forty, when I was a professor at Oxford. Since then I have suffered from that steady deterioration which is the common fate of elderly men and particularly of elderly mathematicians. A mathematician may still be competent enough at sixty, but it is useless to expect him to have original ideas.

It is plain now that my life, for what it is worth, is finished, and that nothing I can do can perceptibly increase or diminish its value. It is very difficult to be dispassionate, but I count it a "success"; I have had more reward and not less than was due to a man of my particular grade of ability. I have held a series of comfortable and "dignified" positions. I have had very little trouble with the duller routine of universities. I hate "teaching", and have had to do very little, such teaching as I have done been almost entirely supervision of research; I love lecturing and have lectured a great deal to extremely able classes; and I have always had plenty of leisure for the research which has been the one great permanent happiness of my life. I have found it easy to work with others and have collaborated on a large scale with two exceptional mathematicians; and this has enabled me to add to mathematics a good deal more than I could reasonably have expected. I have had my disappointments, like any other mathematician, but none of them has been too serious or has made me particularly unhappy. If I had been offered a life neither better nor worse when I was twenty, I would have accepted without hesitation.

It seems absurd to suppose that I could have "done better". I have no linguistic or artistic ability, and very little interest in experimental science. I might have been a tolerable philosopher, but not one of a very original kind. I think that I might have made a good lawyer; but journalism is the only profession, outside academic life, in which I should have felt really confident of my changes. There is no doubt that I was right to be a mathematician, if the criterion is to be what is commonly called success.

My choice was right, then, if what I wanted was a reasonably comfortable and happy life. But solicitors and stockbrokers and bookmakers often lead comfortable and happy lives, and it is very difficult to see how the world is richer for their existence. Is there any sense in which I can claim that my life has been less futile than theirs? It seems to me again that there is only one possible answer: yes, perhaps, but, if so, for one reason only:

I have never done anything "useful". No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world. I have helped to train other mathematicians, but mathematicians of the same kind as myself, and their work has been, so far at any rate as I have helped them to it, as useless as my own. Judged by all practical standards, the value of my mathematical life is nil; and outside mathematics it is trivial anyhow. I have just one chance of escaping a verdict of complete triviality, that I may be judged to have created something worth creating. And that I have created is undeniable: the question is about its value.

The case for my life, then, or for that of any one else who has been a mathematician in the same sense which I have been one, is this: that I have added something to knowledge, and helped others to add more; and that these somethings have a value which differs in degree only, and not in kind, from that of the creations of the great mathematicians, or of any of the other artists, great or small, who have left some kind of memorial behind them.

(from "A Mathematician's Apology," by G. H. Hardy, 1940)

Activity 224. In groups, write an essay "A Message in a Bottle to Fellow Mathematicians" collating your thoughts and experiences, advice and lessons that you wish to pass down to future generations of mathematicians. Exchange your essays with other groups.

Activity 225. "Mathematicians are born, not made" (Henri Poincaré). Make a list of arguments either supporting or opposing the statement. Debate in groups.

Activity 226. In pairs or groups, consult Appendix III and choose two or three mathematicians. Dramatize a talk-show interview between the chosen mathematicians, discussing "your" lives and work.

Unit 30. Mathematics in Fiction

Activity 227. In pairs, discuss the questions.

1. Are you a creative person?
2. Does a mathematician need to be creative? Why?
3. Is mathematics an art form? Why?
4. Do you think the fields of language arts and mathematics are related? If so, how?

Activity 228. In groups, comment on the quotes.

1. “A mathematical equation stands forever.” (Albert Einstein)
2. “Algebra is the intellectual instrument which has been created for rendering clear the quantitative aspect of the world.” (Alfred North Whitehead)
3. “All the effects of nature are only mathematical results of a small number of immutable laws.” (Pierre-Simon Laplace)
4. “Besides language and music, mathematics is one of the primary manifestations of the free creative power of the human mind.” (Hermann Weyl)
5. “Geometry will draw the soul toward truth and create the spirit of philosophy.” (Plato)
6. “In mathematics I can report no deficiency, except it be that men do not sufficiently understand the excellent use of the pure mathematics.” (Francis Bacon)
7. “In mathematics the art of proposing a question must be held of higher value than solving it.” (Georg Cantor)
8. “In mathematics you don't understand things. You just get used to them.” (John von Neumann)
9. “It is impossible to be a mathematician without being a poet in soul.” (Sofya Kovalevskaya)
10. “Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.” (Joseph Fourier)
11. “Mathematics is as much an aspect of culture as it is a collection of algorithms.” (Carl Benjamin Boyer)
12. “Mathematics is no more computation than typing is literature.” (John Allen Paulos)
13. “Mathematics is not only real, but it is the only reality.” (Martin Gardner)
14. “Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.” (David Hilbert)
15. “Mathematics rightly viewed possesses not only truth but supreme beauty.” (Bertrand Russell)
16. “Mighty is geometry; joined with art, resistless.” (Euripides)
17. “Numbers constitute the only universal language.” (Nathanael West)

18. “Probability theory is nothing but common sense reduced to calculation.” (Pierre-Simon Laplace)
19. “Pure mathematics is, in its way, the poetry of logical ideas.” (Albert Einstein)
20. “The highest form of pure thought is in mathematics.” (Plato)
21. “The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.” (G. H. Hardy)
22. “The study of mathematics, like the Nile, begins in minuteness but ends in magnificence.” (Charles Caleb Colton)
23. “When you can measure what you are talking about and express it in numbers, you know something about it.” (Lord Kelvin)
24. “Where there is matter, there is geometry.” (Johannes Kepler)
25. “Wherever there is a number, there is beauty.” (Proclus)

Activity 229. Read the extract from the 2003 novel “The Housekeeper and the Professor” by Japanese writer Yoko Ogawa. Explain the concept of “amicable numbers” based on the text.

"Do you send a lot of articles to magazines?" I asked.

"I wouldn't call them 'articles.' They're just puzzles for amateur mathematicians. Sometimes there's even a prize. Wealthy men who love mathematics put up the money." He looked down, checking his suit in various places, and his gaze fell on a note clipped to his left pocket. "Oh, I see. I sent a proof to the Journal of Mathematics today."

It had been much more than eighty minutes since I'd made my trip to the post office.

"Oh, dear!" I said. "If it's a contest, I should have sent it express mail. If it doesn't get there first, I suppose you don't get the prize."

"No, there was no need to send it express. Of course, it's important to arrive at the correct answer before anyone else, but it's just as important that the proof is elegant."

"I had no idea a proof could be beautiful... or ugly."

"Of course, it can," he said. Getting up from the table, he came over to the sink where I was washing the dishes and peered at me as he continued. "The truly correct proof is one that strikes a harmonious balance between strength and flexibility. There are plenty of proofs that are technically correct but are messy and inelegant or counterintuitive. But it's not something you can put into words — explaining why a formula is beautiful is like trying to explain why the stars are beautiful."

I stopped washing and nodded, not wanting to interrupt the Professor's first real attempt at conversation.

"Your birthday is February twentieth. Two twenty. Can I show you something? This was a prize I won for my thesis on transcendent number theory when I was at college." He took off his wristwatch and held it up for me to see. It was a stylish foreign brand, quite out of keeping with the Professor's rumpled appearance.

"It's a wonderful prize," I said.

"But can you see the number engraved here?" The inscription on the back of the case read President's Prize No. 284.

"Does that mean that it was the two hundred and eighty-fourth prize awarded?"

"I suppose so, but the interesting part is the number 284 itself. Take a break from the dishes for a moment and think about these two numbers: 220 and 284. Do they mean anything to you?"

Pulling me by my apron strings, he sat me down at the table and produced a pencil stub from his pocket. On the back of an advertising insert, he wrote the two numbers, separated strangely on the card.

220

284

"Well, what do you make of them?"

I wiped my hands on my apron, feeling awkward, as the Professor looked at me expectantly. I wanted to respond but had no idea what sort of answer would please a mathematician. To me, they were just numbers.

"Well...," I stammered. "I suppose you could say they're both three-digit numbers. And that they're fairly similar in size — for example, if I were in the meat section at the supermarket, there'd be very little difference between a package of sausage that weighed 220 grams and one that weighed 284 grams. They're so close that I would just buy the one that was fresher. They seem pretty much the same — they're both in the two hundreds, and they're both even —"

"Good!" he almost shouted, shaking the leather strap of his watch. I didn't know what to say. "It's important to use your intuition. You swoop down on the numbers, like a kingfisher catching the glint of sunlight on the fish's fin." He pulled up a chair, as if wanting to be closer to the numbers. The musty paper smell from the study clung to the Professor.

"You know what a factor is, don't you?"

"I think so. I'm sure I learned about them at some point..."

"For 220 is divisible by 1 and by 220 itself, with nothing leftover. So, 1 and 220 are factors of 220. Natural numbers always have 1 and the number itself as factors. But what else can you divide it by?"

"By 2, and 10..."

"Exactly! So, let's try writing out the factors of 220 and 284, excluding the numbers themselves. Like this."

220 : 1 2 4 5 10 11 20 22 44 55 110

142 71 4 2 1 : 284

The Professor's figures, rounded and slanting slightly to one side, were surrounded by black smears where the pencil had smudged.

"Did you figure out all the factors in your head?" I asked.

"I don't have to calculate them — they just come to me from the same kind of intuition you used. So then, let's move on to the next step," he said, adding symbols to the lists of factors.

220 : 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 =

= 142 + 71 + 4 + 2 + 1 : 284

"Add them up," he said. "Take your time. There's no hurry."

He handed me the pencil, and I did the calculation in the space that was left on the advertisement. His tone was kind and full of expectation, and it didn't seem as though he were testing me. On the contrary, he made me feel as though I were on an important mission, that I was the only one who could lead us out of this puzzle and find the correct answer.

I checked my calculations three times to be sure I hadn't made a mistake. At some point, while we'd been talking, the sun had set, and night was falling. From time to time, I heard water dripping from the dishes I had left in the sink. The Professor stood close by, watching me.

"There," I said. "I'm done."

220 : 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284

220 = 142 + 71 + 4 + 2 + 1 : 284

"That's right! The sum of the factors of 220 is 284, and the sum of the factors of 284 is 220. They're called 'amicable numbers,' and they're extremely rare. Fermat and Descartes were only able to find one pair each. They're linked to each other by some divine scheme, and how incredible that your birthday and this number on my watch should be just such a pair."

We sat staring at the advertisement for a long time. With my finger I traced the trail of numbers from the ones the Professor had written to the ones I'd added, and they all seemed to flow together, as if we'd been connecting up the constellations in the night sky.

(from “The Housekeeper and the Professor,”

written by Yoko Ogawa, translated by Stephen Snyder,

2003)

Activity 230. Look at the list of mathematical concepts encountered in the novel “The Housekeeper and the Professor”. Choose one and expand on it.

1. abundant number
2. amicable number
3. Artin’s conjecture
4. deficient number
5. Euler’s formula
6. factorial
7. Fermat’s Last Theorem
8. imaginary number
9. Mersenne prime
10. Napier’s constant
11. perfect number
12. prime number
13. root
14. Ruth-Aaron pair
15. triangular number
16. twin prime

Activity 231. Consult Appendix IV. Choose one film to watch a trailer of. In pairs or groups, role-play a scene from the trailer.

Activity 232. Choose one film from Appendix IV to watch. Write a review of the film. Include a summary of the plot, your impression of the film, and critical evaluation.

Quiz. Mathematical Terminology

1. The __________ of whole numbers is {0, 1, 2, 3, 4, 5, …}.

a. group b. set c. system

2. A number is written using the __________ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

a. digits b. numbers c. numerals

3. The position of a digit in a number determines its __________.

a. magnitude b. place c. value

4. {..., –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5, ...} is called the set of __________.

a. integers b. natural numbers c. whole numbers

5. __________ of whole numbers is combining sets of similar objects.

a. addition b. division c. subtraction

6. __________ of whole numbers is taking away objects from a set.

a. addition b. multiplication c. subtraction

7. __________ property of addition means the order in which whole numbers are added does not change their sum.

a. associative b. commutative c. distributive

8. __________ property of addition means the way in which whole numbers are grouped does not change their sum.

a. associative b. commutative c. distributive

9. __________ of whole numbers is repeated addition but with different notation.

a. exponentiation b. extraction c. multiplication

10. __________ of whole numbers is separating a quantity into equal-sized groups.

a. division b. exponentiation c. subtraction

11. __________ property of multiplication refers to the order in which whole numbers are multiplied does not change their product.

a. associative b. commutative c. distributive

12. __________ property of multiplication refers to the way in which whole numbers are grouped does not change their product.

a. associative b. commutative c. distributive

13. Numbers that are multiplied together are called __________.

a. factors b. products c. quotients

14. If a whole number is divisible by 2, it is called an __________ number.

a. even b. integer c. odd

15. If a whole number is not divisible by 2, it is called an __________ number.

a. even b. integer c. odd

16. One number is divisible by another number if, when we divide them, the __________ is 0.

a. difference b. remainder c. reminder

17. A __________ number is a whole number greater than 1 that has only 1 and itself as factors.

a. composite b. prime c. simple

18. The __________ is used to indicate repeated multiplication.

a. base b. exponent c. root

19. The arithmetic mean, or __________, of a set of numbers is a value around which the values of the numbers are grouped.

a. average b. medium c. middle

20. The symbols $<$ and $>$ are __________ symbols.

a. equality b. inequality c. inequation

21. Positive numbers are __________ than 0 and negative numbers are less than 0.

a. greater b. larger c. more

22. A __________ describes the number of equal parts of a whole.

a. fraction b. proportion c. ratio

23. The number above the fraction bar is called the __________, and the number below is called the denominator.

a. number b. numeral c. numerator

24. If the numerator of a fraction is less than its denominator, the fraction is called a(n) __________ fraction.

a. complex b. improper c. proper

25. If the numerator of a fraction is greater than or equal to its denominator, the fraction is called a(n) __________ fraction.

a. improper b. mixed c. proper

26. One number is the __________ of another if their product is 1.

a. inverse b. opposite c. reciprocal

27. A __________ is the sum of a whole number and a proper fraction.

a. decimal b. fraction c. mixed number

28. A __________ fraction is a fraction whose numerator or denominator, or both, contain one or more fractions or mixed numbers.

a. complex b. improper c. mixed

29. A __________ is the quotient of two numbers or the quotient of two quantities that have the same units.

a. percentage b. proportion c. ratio

30. A __________ is a statement that two ratios are equal.

a. decimal b. fraction c. proportion

31. The first and fourth terms of a __________ are called the extremes, and the second and third terms are called the means.

a. fraction b. proportion c. ratio

32. A __________ is one of the most common tools for measuring lengths.

a. compass b. protractor c. ruler

33. A __________ is one of the most common tools for constructing circles.

a. compass b. protractor c. ruler

34. __________ means parts per one hundred.

a. percent b. proportion c. ratio

35. The __________ of a set of values is the sum of the values divided by the number of values in the set.

a. mean b. median c. mode

36. The __________ of a set of values written in increasing order is the middle value.

a. mean b. median c. mode

37. The __________ of a set of values is the single value that occurs most often.

a. mean b. median c. mode

38. The __________ is a letter (or symbol) that stands for a number.

a. constant b. unknown c. variable

39. Numbers, not changing their values, can be called __________.

a. constants b. unknowns c. variables

40. Variables and/or numbers can be combined with the operations of addition, subtraction, multiplication, and division to create algebraic __________.

a. expressions b. equalities c. equations

41. An __________ is a statement indicating that two expressions are equal.

a. equality b. equation c. identity

42. A number that makes an equation a true statement when substituted for the variable is called a __________ of the equation.

a. constant b. known c. solution

43. If three points lie on the same line, they are called __________.

a. collinear b. concurrent c. coplanar

44. A __________ is a part of a line with two endpoints.

a. curve b. ray c. segment

45. Every line segment has a(n) __________, which divides the segment into two parts of equal length.

a. endpoint b. height c. midpoint

46. A __________ is a part of a line with one endpoint.

a. curve b. ray c. segment

47. A(n) __________ is a figure formed by two rays (called sides) with a common endpoint.

a. angle b. corner c. vertex

48. The common endpoint is called the __________ of the angle.

a. corner b. edge c. vertex

49. A __________ is used to find the measure of an angle.

a. compass b. protractor c. ruler

50. One unit of measurement of an angle is the __________.

a. degree b. power c. radian

51. A(n) __________ angle has a measure that is greater than 0° but less than 90°.

a. acute b. obtuse c. right

52. **A(n) __________ angle has a measure that is greater than 90° but less than 180°.**

a. acute b. obtuse c. right

53. **A __________ angle measures 180°.**

a. right b. round c. straight

54. **A __________ angle measures 90°.**

a. right b. round c. straight

55. **Two angles that have the same vertex and are side-by-side are called __________ angles.**

a. adjacent b. alternate c. vertical

56. **When two lines intersect, pairs of nonadjacent angles are called __________ angles.**

a. adjacent b. alternate c. vertical

57. **Vertical angles are __________.**

a. congruent b. same c. similar

58. **If two lines lie in the same plane, they are called __________.**

a. collinear b. concurrent c. coplanar

59. **__________ lines are coplanar lines that do not intersect.**

a. concurrent b. parallel c. perpendicular

60. **__________ lines are lines that intersect and form right angles.**

a. concurrent b. parallel c. perpendicular

61. **A line that intersects two coplanar lines in two distinct (different) points is called a(n) __________.**

a. intersect b. tangent c. transversal

62. **When a transversal intersects two coplanar lines, four pairs of __________ angles are formed.**

a. complementary b. corresponding c. supplementary

63. **When a transversal intersects two coplanar lines, two pairs of __________ angles and two pairs of alternate __________ angles are formed.**

a. corresponding b. interior c. exterior

64. **A __________ is a closed geometric figure with at least three line segments for its sides.**

a. polygon b. polyhedron c. triangle

65. **The points at which the sides intersect are called __________.**

a. corners b. edges c. vertices

66. **A(n) __________ polygon has sides that are all the same length and angles that are all the same measure.**

a. irregular b. regular c. right

67. **A __________ is a polygon with three sides (and three vertices).**

a. rectangle b. triangle c. trigon

68. **The __________ triangle has all sides of equal length.**

a. equilateral b. isosceles c. scalene

69. **The __________ triangle has at least two sides of equal length.**

a. equilateral b. isosceles c. scalene

70. **The __________ triangle has no sides of equal length.**

a. equilateral b. isosceles c. scalene

71. **The __________ triangle has three angles less than 90°.**

a. acute b. obtuse c. right

72. **The __________ triangle has an angle greater than 90°.**

a. acute b. obtuse c. right

73. **The __________ triangle has one angle equal to 90°.**

a. acute b. obtuse c. right

74. **The longest side of a right triangle is called the __________.**

a. cathetus b. hypotenuse c. leg

75. **In a(n) __________ triangle, the angles opposite the sides of equal length are called base angles; the third angle is called the vertex angle.**

a. scalene b. equilateral c. isosceles

76. **The sum of the measures of the angels of a __________ is 180°.**

a. rectangle b. square c. triangle

77. **If two triangles have the same size and the same shape, they are __________ triangles.**

a. congruent b. same c. similar

78. **__________ triangles have the same shape, but not necessarily the same size.**

a. congruent b. same c. similar

79. **A __________ is a polygon with four sides.**

a. polyhedron b. quadrilateral c. tetragon

80. A segment that joins two nonconsecutive vertices of a polygon is called a __________ of the polygon.

a. diagonal b. height c. radius

81. A __________ is a quadrilateral with opposite sides parallel.

a. parallelogram b. trapezium c. trapezoid

82. A __________ is a parallelogram with four right angles.

a. rectangle b. rhombus c. square

83. A __________ is a rectangle with sides of equal length.

a. rhomboid b. rhombus c. square

84. A __________ is a parallelogram with sides of equal length.

a. rectangle b. rhombus c. square

85. A __________ is a quadrilateral with exactly two sides parallel.

a. rhombus b. trapezium c. trapezoid

86. The parallel sides of a __________ are called bases; the nonparallel sides are called legs.

a. tetragon b. trapezium c. trapezoid

87. The __________ of a polygon is the distance around it.

a. area b. circumference c. perimeter

88. The __________ of a polygon is the measure of the amount of surface it encloses.

a. area b. circumference c. perimeter

89. A __________ is the set of all points in a plane that lie a fixed distance from a point called its origin.

a. centre b. circle c. circumference

90. The fixed distance from the origin is the circle's __________.

a. chord b. diameter c. radius

91. A __________ of a circle is a line segment connecting two points on the circle.

a. chord b. diameter c. radius

92. A __________ is a chord that passes through the circle's centre.

a. diagonal b. diameter c. radius

93. Any part of a circle is called a(n) __________.

a. arc b. quarter c. semicircle

94. A __________ is an arc of a circle whose endpoints are the endpoints of a diameter.

a. quarter b. sector c. semicircle

95. The __________ is the perimeter of a circle.

a. area b. circumference c. length

96. Operators separate an algebraic expression into parts called __________.

a. elements b. members c. terms

97. A(n) __________ is a single term or a sum of terms in which all variables have whole-number exponents, and no variable appears in the denominator.

a. coefficient b. expression c. polynomial

98. A polynomial with one term is called a __________.

a. binomial b. monomial c. trinomial

99. A polynomial with two terms is called a __________.

a. binomial b. monomial c. trinomial

100. A polynomial with three terms is called a __________.

a. binomial b. monomial c. trinomial

Appendix I. Cohesive Devices

Sequencing	Segueing
to begin with	as for
to start with	as to
first of all	concerning
first and foremost	as far as … is concerned
first / firstly	where … is concerned
second / secondly	regarding
third / thirdly	as regards
lastly	in regard to
last of all	with regard to
last but not least	relative to
finally	in relation to
next	in respect of
then	with respect to
later	touching
afterward(s)	speaking of
subsequently	talking of
in time	in terms of
over time	
with time	
in the meantime	
meanwhile	
at first	
initially	
originally	
eventually	
ultimately	

Adding	Conceding
and	but
also	yet
plus	still
too	though
as well	although
either	even though
both	even so
both of	however
both … and …	nevertheless
not only … but also	nonetheless
… and … alike	despite
as well as	in spite of
together with	regardless of
along with	irrespective of
alongside	albeit
in addition to	notwithstanding
either … or	in fact
neither … nor	in actual fact
additionally	as a matter of fact
in addition	assuming
besides	provided
moreover	providing
furthermore	
what is more	
by the by(e)	
by the way	
incidentally	
not to mention	
let alone	
to say nothing of	
never mind	
much less	
on top of (it)	
to top it all (off)	

Exemplifying	Explaining
e. g.	i. e.
for example	that is
for instance	that is to say
to exemplify	in other words
to illustrate	according to
like	what I mean is
such as	in theory
namely	theoretically
including …	hypothetically
… included	in principle
in particular	in practice
particularly	in reality
especially	in fact
specifically	in actual fact
not least	as a matter of fact
is a prime example of	the fact of the matter is
is a case in point	in a way
among other things	in a sense
etc	so to say
and so on	so to speak
and so forth	as it were
and so on and so forth	in a manner of speaking
and the like	
and such like	
to name but a few	
to mention but a few	

Expressing Cause	Expressing Effect
because	so
as	thus
since	ergo
for	hence
because of	thereby
as a result of	therefore
due to	accordingly
owing to	consequently
thanks to	in consequence
on account of	as a consequence
on the back of	
in (the) light of	
in view of	
in the wake of	
by virtue of	
courtesy of	
seeing as/that	
considering	
taking into account	
taking into consideration	
for one thing … for another (thing)	

Comparing	Contrasting
like	while
likewise	whereas
similarly	unlike
equally	otherwise
in equal measure	alternatively
in common with	conversely
in line with	on the contrary
compared to/with	out of line with
in/by comparison	instead
in/by comparison to/with	instead of
relative to	except
in relation to	except for
	apart from
	aside from
	other than
	but for
	bar
	save
	save for
	excepting …
	… excepted
	with the exception of
	excluding …
	… excluded
	in/by contrast
	in contradistinction to
	having said that
	that is not to say
	on the one hand … on the other hand
	on one hand … on the other

Emphasizing	Summarizing
even	all in all
not to say	overall
as such	briefly
per se	in brief
only	in short
just	by and large
merely	broadly
simply	broadly speaking
solely	in general
barely	generally
hardly	generally speaking
scarcely	on the whole
almost	to conclude
nearly	in conclusion
practically	in sum
virtually	in summary
all but	to sum up
just about	to summarize
next to	in a word
basically	in one word
fundamentally	to put simply
actually	simply put
in actuality	in a nutshell
effectively	to put it in a nutshell
in effect	long story short
essentially	to cut/make a long story short
in essence	the long and the short of (it)
substantially	suffice to say
in substance	suffice it to say that
no doubt	all things considered
doubtless	the bottom line is
undoubtedly	
undeniably	
indubitably	
unquestionably	
beyond/without (a) doubt	
beyond/without a shadow of a doubt	
needless to say	
it goes without saying	

Expressing Knowledge	Expressing Opinion
if memory serves	Personally…
if my memory serves me right	I think
as far as I recall	I suppose
as far as I can remember	I believe
as far as I know	I guess
as far as I’m aware	I gather
to my knowledge	I assume
to the best of my knowledge	I presume
I know for a fact	I reckon
I’m certain that	I’m inclined to think
I’m confident that	I incline to the thought that
I’m positive that	in my opinion
I don’t know	in my view
I don’t have a clue	in my eyes
I haven’t (got) a clue	in my book
I have (got) no idea/clue	to my mind
I haven’t the faintest idea/notion	to my way of thinking
I haven’t the remotest idea/notion	from where I stand
I haven’t the slightest idea/notion	from my standpoint
	from my viewpoint
	from my point of view
	from my vantage point
	from my perspective
	as far as I’m concerned
	as far as I can see
	as far as I can tell
	it appears to me
	it seems to me
	the way I see it is
	my understanding is
	honestly
	if I’m honest
	to be honest
	in all honesty
	frankly
	to be frank
	let me be frank
	in truth
	truthfully
	(if) truth be told/known
	in all truthfulness
	to tell (you) the truth
	in all sincerity

Agreeing	Disagreeing
absolutely	I'm afraid that…
totally	I'm sorry but…
exactly	I cannot agree
precisely	I disagree about/on/over
certainly	I disagree with
definitely	I somewhat disagree
clearly	I completely disagree
naturally	I entirely disagree
obviously	I totally disagree
of course	let's agree to disagree
right on	let's agree to differ
spot on	I beg to differ
this/that is it	I hear you but
quite (so)	I hear what you're saying but
quite right	I get your point but
very much so	I see your point but
I'm with you	I take your point but
my thoughts exactly	
my sentiments exactly	
one hundred percent	
I agree about/on/upon	
I agree with	
I'm inclined to agree	
I somewhat agree	
I quite agree (BrE "somewhat")	
I quite agree (AmE "completely")	
I completely agree	
I absolutely agree	
I fully agree	
I entirely agree	
I couldn't agree more	
it's difficult/hard to disagree	
I can/could hardly disagree	
you have (got) a point there	

Appendix II. Model Overview

Title **Subtitle**	The title/headline of the article is The article is entitled/headlined The subtitle of the article is The article is subtitled
Author(s)	The author of the article is The article is written by
Source	It was published in/on It was printed in It was posted on
Main Idea **Gist**	The (author of the) article / paper / study / research • is about • is devoted to • is concerned with • looks at/into • deals with • delves into • dwells on • describes, discusses • deliberates on/over/about • goes into detail about • gives/provides an insight into • analyzes, examines, explores, investigates • focuses on (focusses on), concentrates on, touches on • emphasizes, highlights, stresses, underlines, points out • clarifies, explains, elucidates, specifies • elaborates on, expands on, expounds on • presents, shows, depicts, demonstrates • covers, names, lists, enumerates
Sections **Subheadings**	The article consists of/includes/comprises ... parts/sections. Their respective/corresponding subheadings are
Contents **Inferences**	It is abundantly clear from the text that The problem of … is addressed/approached from many different angles.

	The article says/states/reports that
	Special attention is paid to
	Great importance is attached to
	One of the key points to be singled out is
	It is shown/suggested that
	It should be mentioned/noted that
	Among other things, the issue of … is raised.
	There is an assumption/presumption/supposition that
	The text offers abstract/practical/concrete/typical/illustrative examples of
	Some data is/are given to illustrate the fact that
	… is quoted/cited as saying that
	The author takes the position that
	The author comes to/arrives at/reaches the conclusion that
Message	The article paints a clear/full/accurate/vague picture of
Opinion	The article reflects favourable/ambivalent/changing attitudes to/towards
	The idea of … appears/seems brilliant/valuable/innovative/questionable.
	The issue of … is topical/relevant.
	The statement expressed/voiced by … could be interpreted as
	The plan/project/solution is realistic/ingenious/effective.
	The author’s ideas on the subject of … are (not) the ones I share/support.
	I found the article informative/controversial/complicated/inconsistent.

Appendix III. Prominent Mathematicians

Russian (Soviet)	Nikolai Ivanovich Lobachevsky (1792–1856)
	Mikhail Vasilyevich Ostrogradsky (1801–1862)
	Pafnuty Lvovich Chebyshev (1821–1894)
	Sofya Vasilyevna Kovalevskaya (1850–1891)
	Andrey Andreyevich Markov (1856–1922)
	Aleksandr Mikhailovich Lyapunov (1857–1918)
	Andrey Nikolaevich Kolmogorov (1903–1987)
	Sergei Lvovich Sobolev (1908–1989)
	Israel Moiseevich Gelfand (1913–2009)
	Grigori Yakovlevich Perelman (born 1966)
Greek	Thales of Miletus (ca. 625–547 B.C.E.)
	Anaximander (ca. 610–546 B.C.E.)
	Pythagoras of Samos (ca. 569–490 B.C.E.)
	Hippocrates of Chios (ca. 470–410 B.C.E.)
	Plato (ca. 428–348 B.C.E.)
	Eudoxus of Cnidus (ca. 408–355 B.C.E.)
	Aristotle (384–322 B.C.E.)
	Euclid (ca. 300 B.C.E.)
	Archimedes of Syracuse (287–212 B.C.E.)
	Eratosthenes of Cyrene (ca. 275–195 B.C.E.)
	Apollonius of Perga (ca. 240–190 B.C.E.)
	Hero (Heron) of Alexandria (ca. 60 C.E.)
	Menelaus of Alexandria (ca. 100 C.E.)
	Claudius Ptolemy (ca. 100–170 C.E.)
	Diophantus of Alexandria (ca. 200–284 C.E.)
	Pappus of Alexandria (ca. 320 C.E.)
	Hypatia (ca. 355–415 C.E.)

Indian	Aryabhata (ca. 476–550) Brahmagupta (ca. 598–665) Bhaskara II (Bhaskaracharya) (1114–1185) Madhava of Sangamagramma (ca. 1350–1425) Srinivasa Ramanujan (1887–1920)
Persian (Iranian)	Muhammad ibn Musa al-Khwarizmi (ca. 780–850) Omar Khayyam (1048–1131) Jamshid Mas'ud al-Kashi (ca. 1380–1429)
Chinese	Zu Chongzhi (ca. 429–500) Li Ye (1192–1279) Qin Jiushao (1202–1261) Zhu Shijie (ca. 1280–1303)
Italian	Fibonacci (Leonardo Fibonacci, Leonardo of Pisa) (ca. 1170–1250) Scipione del Ferro (1465–1526) Niccolò Fontana Tartaglia (1499–1557) Girolamo Cardano (1501–1576) Lodovico de Ferrari (1522–1565) Galileo Galilei (1564–1642) Bonaventura Francesco Cavalieri (1598–1647) Evangelista Torricelli (1608–1647) Giovanni Girolamo Saccheri (1667–1733) Maria Gaetana Agnesi (1718–1799)
French	Nicole Oresme (ca. 1323–1382) François Viète (1540–1603) Girard Desargues (1591–1661) René Descartes (1596–1650) Pierre de Fermat (1601–1665) Blaise Pascal (1623–1662) Abraham De Moivre (1667–1754)

	Pierre Louis Moreau de Maupertuis (1698–1759)
	Jean-Baptiste Le Rond d’Alembert (1717–1783)
	Joseph-Louis Lagrange (1736–1813)
	Pierre-Simon, marquis de Laplace (1749–1827)
	Adrien-Marie Legendre (1752–1833)
	Jean Baptiste Joseph Fourier (1768–1830)
	André Marie Ampère (1775–1836)
	Siméon Denis Poisson (1781–1840)
	Augustin-Louis Cauchy (1789–1857)
	Évariste Galois (1811–1832)
	Jules-Henri Poincaré (1854–1912)
German	Regiomontanus (1436–1476)
	Johannes Kepler (1571–1630)
	Gottfried Wilhelm Leibniz (1646–1716)
	Carl Friedrich Gauss (1777–1855)
	Friedrich Wilhelm Bessel (1784–1846)
	Karl Gustav Jacob Jacobi (1804–1851)
	Peter Gustav Lejeune Dirichlet (1805–1859)
	Karl Theodor Wilhelm Weierstrass (1815–1897)
	Rudolf Julius Emanuel Clausius (1822–1888)
	Leopold Kronecker (1823–1891)
	Georg Friedrich Bernhard Riemann (1826–1866)
	Julius Wilhelm Richard Dedekind (1831–1916)
	Georg Cantor (1845–1918)
	Christian Felix Klein (1849–1925)
	David Hilbert (1862–1943)
	Hermann Minkowski (1864–1909)
	Ernst Friedrich Ferdinand Zermelo (1871–1953)
	Albert Einstein (1879–1955)

	Amalie Emmy Noether (1882–1935) Hermann Klaus Hugo Weyl (1885–1955)
Swiss	Jakob Bernoulli (1655–1705) Johann Bernoulli (1667–1748) Daniel Bernoulli (1700–1782) Leonhard Euler (1707–1783) Johann Heinrich Lambert (1728–1777) Johann Jakob Balmer (1825–1898)
British	Robert Recorde (ca. 1510–1558) John Dee (1527–1608) John Napier (1550–1617) Henry Briggs (1561–1630) William Oughtred (1574–1660) Edmund Gunter (1581–1626) John Wallis (1616–1703) Isaac Barrow (1630–1677) Sir Isaac Newton (1642–1727) Edmond Halley (1656–1742) Brook Taylor (1685–1731) Colin Maclaurin (1698–1746) Charles Babbage (1791–1871) Augustus de Morgan (1806–1871) Ada Lovelace (1815–1852) George Boole (1815–1864) Arthur Cayley (1821–1895) Sir George Howard Darwin (1845–1912) Karl Pearson (1857–1936) Alfred North Whitehead (1861–1947) Bertrand Arthur William Russell (1872–1970)

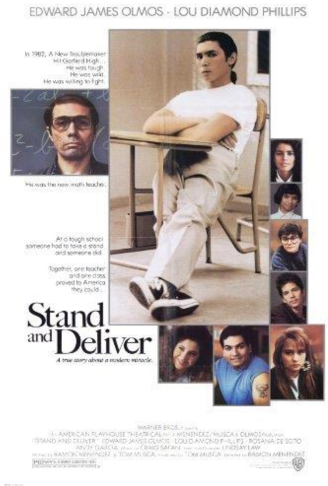
	Sir Ronald Aylmer Fisher (1890–1962)
	William George Penney (1909–1991)
	Alan Mathison Turing (1912–1954)
	Sir Hermann Bondi (1919–2005)
	Sir Roger Penrose (born 1931)
Irish	Sir William Rowan Hamilton (1805–1865)
American	Josiah Willard Gibbs (1839–1903)
	Charles Sanders Peirce (1839–1914)
	Norbert Wiener (1894–1964)
	Howard Hathaway Aiken (1900–1973)
	John von Neumann (1903–1957)
	Kurt Gödel (1906–1978)
	Claude Elwood Shannon (1916–2001)
Hungarian	János Bolyai (1802–1860)
Russian	Nikolai Ivanovich Lobachevsky (1792–1856)
	Pafnuty Lvovich Chebyshev (1821–1894)
	Sofya Vasilyevna Kovalevskaya (1850–1891)
	Andrey Nikolaevich Kolmogorov (1903–1987)

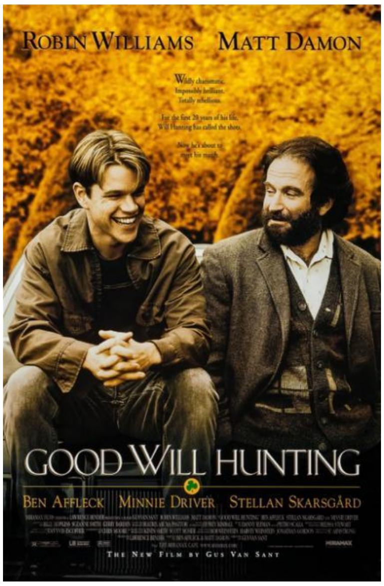
Appendix IV. Mathematics in Film

Stand and Deliver (1988) [American]

Drama, True Story

Jaime Escalante is a mathematics teacher in a school in a Hispanic neighbourhood. Convinced that his students have potential, he adopts unconventional teaching methods to help gang members and no-hopers pass the rigorous Advanced Placement exam in calculus.





Good Will Hunting (1997) [American]

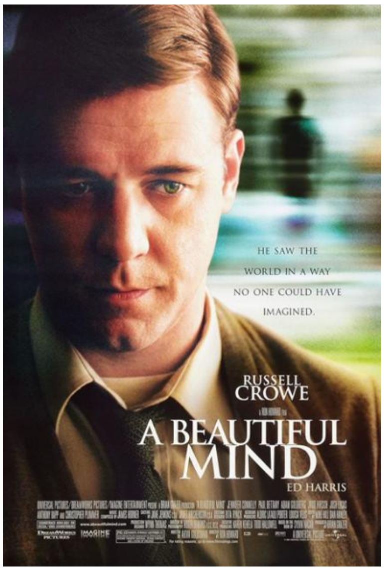
Drama

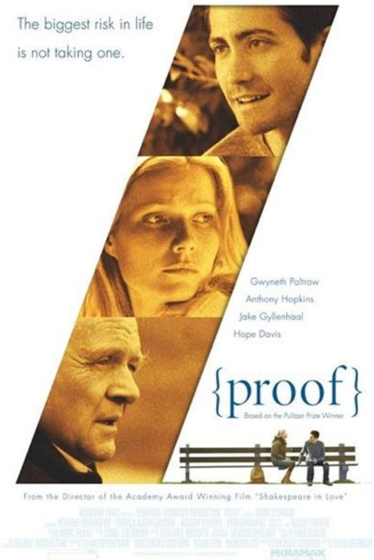
A young adult by the name of Will Hunting has always been living in the slums. He spends his nights at bars and batting cages with his best friends. However, he has an amazing talent in mathematics. When Will is almost sent to jail, a professor at MIT decides to bail him out under the condition that he works with him for math every week and that he visits a therapist. Little does he know that the therapist, Sean, will change his life in so many ways.

A Beautiful Mind (2001) [American]

Drama, True Story, Book Adaptation

At Princeton University, John Nash struggles to make a worthwhile contribution to serve as his legacy to the world of mathematics when he finally makes a revolutionary breakthrough. After graduate school he turns to teaching. Meanwhile the government asks his help with breaking Soviet codes, which soon gets him involved in a terrifying conspiracy plot. Nash grows more and more paranoid until a discovery that turns his entire world upside down.





Proof (2005) [American]

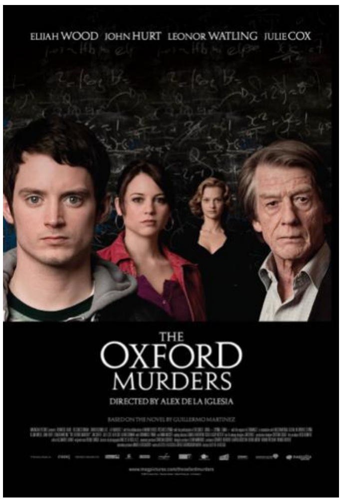
Drama, Play Adaptation

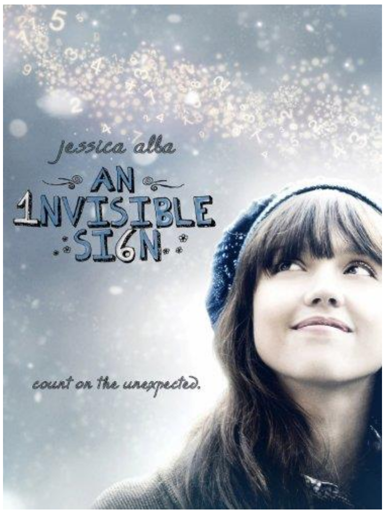
Catherine is a woman in her late twenties who is strongly devoted to her father, Robert, a brilliant and well-known mathematician whose grip on reality is beginning to slip away. As Robert descends into madness, Catherine begins to wonder if she may have inherited her father's mental illness along with his mathematical genius. When Robert's work reveals a mathematical proof of potentially historic proportions, it sets off shock waves in more ways than one.

The Oxford Murders (2008) [British]

Mystery, Book Adaptation

At Oxford University, a professor and a graduate student work together to try to stop a potential series of murders seemingly linked by mathematical symbols.





An Invisible Sign (2010) [American]

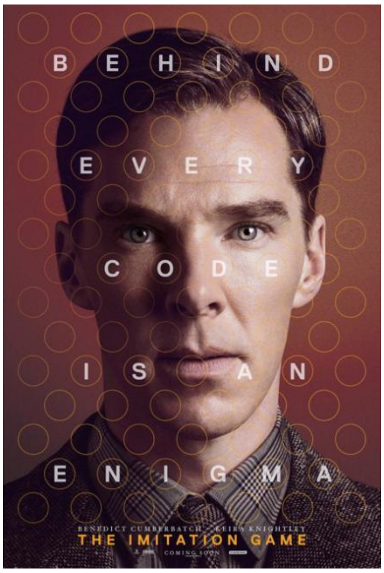
Drama, Book Adaptation

Mona Gray is a 20-year-old loner who turned to math for salvation as a child after her father became ill. As an adult she now teaches the subject and helps her students through their own crises.

X+Y / A Brilliant Young Mind (2014) [British]

Drama

In a world difficult to comprehend, Nathan struggles to connect with those around him — most of all his loving mother — but finds comfort in numbers. When Nathan is taken under the wing of unconventional and anarchic teacher, Mr. Humphreys, the pair forge an unusual friendship and Nathan's talents win him a place on the UK team at the International Mathematics Olympiad. From suburban England to bustling Taipei and back again, Nathan builds complex relationships as he is confronted by the irrational nature of love.



The Imitation Game (2014) [British]

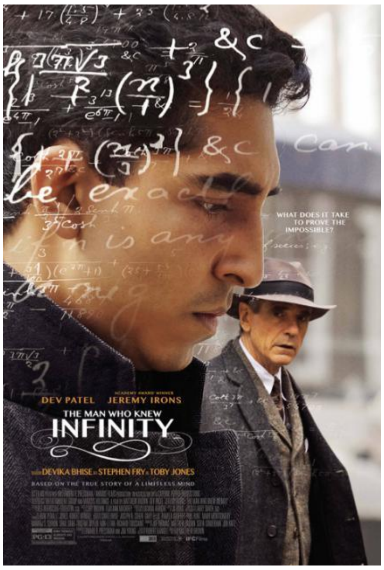
Drama, True Story, Book Adaptation

In 1939, newly created British intelligence agency MI6 recruits Cambridge mathematics alumnus Alan Turing to crack Nazi codes, including Enigma — which cryptanalysts had thought unbreakable. Turing's team analyze Enigma messages while he builds a machine to decipher them.

The Man Who Knew Infinity (2015) [British]

Drama, True Story, Book Adaptation

The story of the life and academic career of the pioneer Indian mathematician, Srinivasa Ramanujan, and his friendship with his mentor, Professor G. H. Hardy.



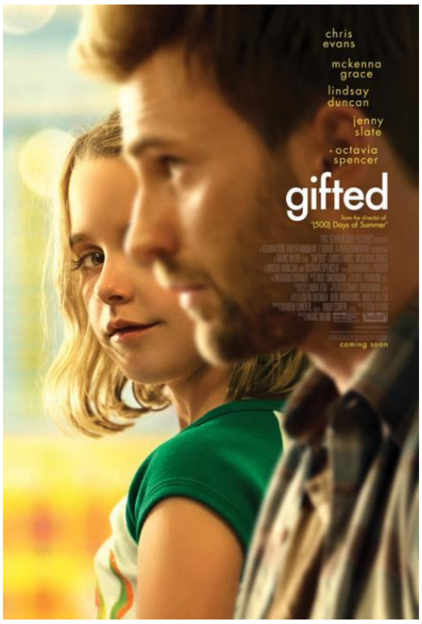
Hidden Figures (2016) [American]

Drama, True Story, Book Adaptation

The story of a team of female African-American mathematicians who served a vital role in NASA during the early years of the US space program.

Gifted (2017) [American]

Drama



Frank Adler is a single man raising a child prodigy — his spirited young niece Mary in a coastal town in Florida. Frank’s plans for a normal school life for Mary are foiled when the seven-year-old’s mathematical abilities come to the attention of Frank’s formidable mother Evelyn whose plans for her granddaughter threaten to separate Frank and Mary.

Appendix V. Mathematical Jokes and Puns

1. 60 degrees is such A CUTE angle.
2. 7 was standing on the shoulders of 5 and fell off. You know why? Because that is so IMPROPER.
3. A decimal number told a whole number: “There is no POINT in talking to you.”
4. A kid said to his math teacher: “To show you how good I am at fractions, I only did HALF my homework.”
5. A student asked their teacher if they would have any problems on the upcoming test. The teacher replied, “I think you’ll have lots of PROBLEMS on the test.”
6. A student turned in a blank sheet of paper for his math test, and the teacher asked him why. “It was on IMAGINARY numbers,” he said. “Can’t you see them?”
7. Are monsters good at math? Not unless you COUNT Dracula.
8. Are you cold? Well, then go to the corner of the room where it's 90 DEGREES.
9. Dear algebra, stop trying to find your X. They’re never coming back — don’t ask Y.
10. Did you hear about the over-educated circle? It has 360 DEGREES.
11. Did you hear the joke about the statistician? PROBABLY.
12. Did you know this nautical fact? 3.14% of sailors are PIrates.
13. Do you know what’s ODD? Every other number.
14. How do mathematicians reprimand their kids? If I've told you N times, I’ll tell you N+1 times.
15. How do you get from point A to point B? Just take an x-y PLANE or a rhomBUS.
16. How do you make SEVEN an EVEN number? Remove the S.
17. How many bakers does it take to make a PIe? 3.14.
18. I don’t get the POINT of decimals. I’m more partial to fractions.
19. I had an argument with a 90-degree angle. It turns out it was RIGHT.
20. I knew a mathematician who couldn’t afford lunch. He could BUYnomials.
21. I would tell you a joke about an infinite line. But it doesn’t have an ENDPOINT.
22. I’ll do algebra, and I’ll do trig. I’ll even do statistics. But graphing is where I DRAW THE LINE.
23. I’ve decided to become a math teacher, but I’m only going to teach subtraction. I just want to MAKE A DIFFERENCE.
24. Is it true that old mathematicians never die? Yes, they just lose some of their FUNCTIONS.
25. It’s always a good idea to bring a mathematician camping. They come prepared with a pair of AXES.
26. It’s so sad to think that parallel lines have so much in common. But they’ll never be able to MEET.
27. Ladies and gentlemen, my next song is entitled “Subtraction”. TAKE IT AWAY.
28. Mathematician: “πr^2”. Baker: “No, PIES are round, and cakes are SQUARE.”
29. Pi was fighting with an imaginary number: “Get REAL,” pi said. “Be RATIONAL,” the imaginary number said.
30. 7 asked 9, “Looks like you have put on some weight?” 9 replied, “Yeah, I ROUNDED UP.”
31. Teacher: “Why are you doing your multiplication on the floor?” Student: “You told me not to use TABLES.”
32. The minus sign tried to explain to the plus sign how multiplication works, but he only understood SUM of it.

33. The minus sign was talking to the positive sign. The minus sign asked, “Are you sure I make a DIFFERENCE?” and the other sign said, “I’m POSITIVE.”
34. The problem with math puns is that calculus jokes are all DERIVATIVE, trigonometry jokes are too GRAPHIC, algebra jokes are usually FORMULAIC, and arithmetic jokes are pretty BASIC. But I guess the occasional statistics joke is an OUTLIER.
35. There are THREE kinds of people in the world. Those who can count and those who can’t.
36. There is a fine LINE between a numerator and a denominator. But only a fraction would understand.
37. What are ten things you can always COUNT ON? Your fingers.
38. What did one algebra book say to the other? Don’t bother me, I’ve got my own PROBLEMS.
39. What did pi say in a fight with its brother? You’re being IRRATIONAL.
40. What did the calculator say to the student? You can always COUNT ON me.
41. What did the student say about the equation she couldn’t solve? This is DERIVEing me crazy.
42. What did you think of the film American PIe? Meh, I give it 3.14 stars.
43. What do baby parabolas drink? Quadratic FORMULA.
44. What do mathematicians and the Air Force have in common? They both use PI-lots.
45. What do mathematicians do after a snowstorm? Make snow ANGLES.
46. What do you call a wrecked angle? A RECTangle.
47. What do you call a man who spent all summer at the beach? A TAN-GENT.
48. What do you call people who are in favour of tractors? PRO-TRACTORS.
49. What do you call the number 7 and the number 3 who got married? The ODD couple.
50. What do you get when you divide the circumference of the sun by its diameter? PIe in the sky.
51. What do you get when you multiply a New York City landmark by itself? TIMES SQUARE.
52. What don’t atheists do well with exponents? Because they don’t believe in higher POWERS.
53. What geometric shape removes curses? A HEXaGON.
54. What happens when you keep missing math class? It really starts to ADD UP.
55. What is 2n plus 2n? I don't know. It sounds 4N to me.
56. What is the solution to any equation? Multiply both sides by 0.
57. What shape do you always have to be careful of? A TRAPezoid.
58. What shape is usually waiting for you inside a Starbucks? A LINE.
59. What's a math teacher's favourite dessert? A PIe.
60. What’s a math teacher’s favourite kind of tree? GeomeTREE.
61. What’s a math teacher’s favourite season? SUMmer.
62. What’s a swimmer's favourite kind of math? DIVEision.
63. What’s the best way to get a math tutor? An AD.
64. What’s the most adventurous type of number? ROAMIN’ numerals.
65. What’s the official animal of PI Day? The PIthon.
66. Where did the geometry teacher go on holiday? Who knows? All I know is that she’s polyGON.
67. Which king loved fractions? Henry the 1/8.
68. Which snakes are good at math? ADDers.
69. Which tool is best for math? The multiPLIERS.
70. Who invented the Round Table? Sir Cumference.
71. Who’s the king of the pencil case? The RULER.
72. Why are 60 degrees and 30 degrees proud of their child 90 degrees? Because it is always RIGHT.

73. Why are obtuse angles always so sad? They're never RIGHT.
74. Why can't you trust a math teacher holding graphing paper? They must be PLOTTING something.
75. Why can't you trust a math teacher? They're always CALCULATING.
76. Why couldn't the angle get a loan? Its parents wouldn't CO-SINE.
77. Why did John have trouble memorizing multiplication tables? Because TIMES were difficult.
78. Why did Pi fail its driver's test? Because it didn't know when to STOP.
79. Why did the 30-60-90 triangle marry the 45-45-90 triangle? They were RIGHT for each other.
80. Why did the mathematician spill all of his food in the oven? The directions said, "Put it in the oven at 180 DEGREES".
81. Why did the obtuse angle jump in the pool? Because it was over 90 DEGREES.
82. Why did the student wear glasses in math class? To improve diVISION.
83. Why did the two fours skip lunch? They already EIGHT.
84. Why do mathematicians like parks? Because of all the NATURAL LOGS.
85. Why do plants hate math? Because it gives them SQUARE ROOTS.
86. Why don't broken calculators have friends? Because you can't COUNT ON them.
87. Why is math considered to be codependent? It relies on others to solve its PROBLEMS.
88. Why is six afraid of seven? Because seven EIGHT nine.
89. Why is statistics never anyone's favourite subject? It's just AVERAGE.
90. Why should you never fight with decimals? They always have a POINT.
91. Why shouldn't you be afraid of advanced math? Because it is AS EASY AS PIe.
92. Why was 10 very happy when 2 was not around? Because 10 finally EIGHT.
93. Why was algebra so easy for the Romans? X was always 10.
94. Why was math class so long? The teacher kept GOING OFF ON A TANGENT.
95. Why was Mr. Smith's class so noisy? He liked to practice GONG DIVISION.
96. Why was the equal sign so humble? Because she knew she wasn't GREATER THAN or LESS THAN anyone else.
97. Why was the math book sad? Because it had so many PROBLEMS.
98. Why was the student confused when he went from English class to math class? Because he was taught that a DOUBLE NEGATIVE in English is bad, but in math, it's a POSITIVE.
99. You know what seems ODD to me? Numbers that can't be divided by two.
100. You should never start a conversation with Pi. It'll just go on and on FOREVER.

Appendix VI. Glossary of Mathematical Terms

Term	*Translation*	*Definition*
acute angle	острый угол	an angle that is less than 90°
addition	сложение	a mathematical operation in which the sum of two numbers or quantities is calculated
angle	угол	the extent to which one such line or plane diverges from another, measured in degrees or radians
arc	дуга	a section of a curve, graph, or geometric figure
area	площадь	the two-dimensional extent of the surface of a solid, or of some part thereof, esp. one bounded by a closed curve
average	среднее значение	the result obtained by adding the numbers or quantities in a set and dividing the total by the number of members in the set
axis	ось	one of two or three reference lines used in coordinate geometry to locate a point in a plane or in space
base	основание	the number of distinct single-digit numbers in a counting system
binary	двоичный	of, relating to, or expressed in binary notation or binary code
binomial	двучлен	a mathematical expression consisting of two terms
cardinal number	кардинальное число	a number denoting quantity but not order in a set
Cartesian coordinates	декартовы координаты	a system of representing points in space in terms of their distance from a given origin measured along a set of mutually perpendicular axes
chord	хорда	a straight line connecting two points on a curve or curved surface
circle	окружность	a closed plane curve every point of which is equidistant from a given fixed point, the centre
circumference	длина окружности	the boundary of a specific area or geometric figure, esp. of a circle
closed set	замкнутое множество	a set that includes all the values obtained by application of a given operation to its members
coefficient	коэффициент	a numerical or constant factor in an algebraic term
common denominator	общий знаменатель	an integer exactly divisible by each denominator of a group of fractions
common factor	общий множитель	a number or quantity that is a factor of each member of a group of numbers or quantities

Term	*Translation*	*Definition*
complex number	комплексное число	any number of the form a + ib, where “a” and “b” are real numbers and i = $\sqrt{-1}$
concentric	концентрический	having a common centre
cone	конус	a geometric solid consisting of a plane base bounded by a closed curve, often a circle or an ellipse, every point of which is joined to a fixed point, the vertex, lying outside the plane of the base
constant	постоянная	a symbol representing an unspecified number that remains invariable throughout a particular series of operations
coordinate	координата	any of a set of numbers that defines the location of a point in space
cosecant	косеканс	a trigonometric function that in a right-angled triangle is the ratio of the length of the hypotenuse to that of the opposite side; the reciprocal of sine
cosine	косинус	a trigonometric function that in a right-angled triangle is the ratio of the length of the adjacent side to that of the hypotenuse; the sine of the complement
cotangent	котангенс	a trigonometric function that in a right-angled triangle is the ratio of the length of the adjacent side to that of the opposite side; the reciprocal of tangent
cube	куб	a solid having six plane square faces in which the angle between two adjacent sides is a right angle
cube root	кубический корень	the number or quantity whose cube is a given number or quantity
cuboid	прямоугольный параллелепипед	a geometric solid whose six faces are rectangles; rectangular parallelepiped
curve	кривая	a system of points whose coordinates satisfy a given equation; a locus of points
cylinder	цилиндр	a solid consisting of two parallel planes bounded by identical closed curves, usually circles, that are interconnected at every point by a set of parallel lines, usually perpendicular to the planes
decagon	десятиугольник	a polygon having ten sides
decimal	десятичная дробь	a fraction that has a denominator of a power of ten, the power depending on or deciding the decimal place
denary	десятичный	calculated by tens; based on ten; decimal
denominator	знаменатель	the divisor of a fraction
diagonal	диагональ	any oblique row of squares of the same colour

Term	*Translation*	*Definition*
diameter	диаметр	a straight line connecting the centre of a geometric figure, esp. a circle or sphere, with two points on the perimeter or surface
digit	цифра	any of the ten Hindu-Arabic numerals from 0 to 9
division	деление	a mathematical operation, the inverse of multiplication, in which the quotient of two numbers or quantities is calculated
dodecahedron	додекаэдр	a solid figure having twelve plane faces
equal(s) sign	знак равенства	the sign =, which is used in arithmetic to indicate that two numbers or sets of numbers are equal
ellipse	эллипс	a closed conic section shaped like a flattened circle and formed by an inclined plane that does not cut the base of the cone
equation	уравнение	a mathematical statement that two expressions are equal: it is either an identity in which the variables can assume any value, or a conditional equation in which the variables have only certain values (roots)
equilateral	равносторонний	a geometric figure having all its sides of equal length
even	четный	(of a number) divisible by two
factor	множитель	one of two or more integers or polynomials whose product is a given integer or polynomial
factorial	факториал	the product of all the positive integers from one up to and including a given integer
formula	формула	a general relationship, principle, or rule stated, often as an equation, in the form of symbols
fraction	дробь	a ratio of two expressions or numbers other than zero
frequency	частота	the number of times that an event occurs within a given period; rate of recurrence
function	функция	a relation between two sets that associates a unique element (the value) of the second (the range) with each element (the argument) of the first (the domain)
graph	график	a drawing depicting the relation between certain sets of numbers or quantities by means of a series of dots, lines, etc, plotted with reference to a set of axes
helix	винтовая линия	a curve that lies on a cylinder or cone, at a constant angle to the line segments making up the surface; spiral
hemisphere	полусфера	one half of a sphere
heptagon	семиугольник	a polygon having seven sides

Term	*Translation*	*Definition*
hexagon	шестиугольник	a polygon having six sides
hyperbola	гипербола	a conic section formed by a plane that cuts both bases of a cone; it consists of two branches asymptotic to two intersecting fixed lines and has two foci
hypotenuse	гипотенуза	the side in a right-angled triangle that is opposite the right angle
icosahedron	икосаэдр	a solid figure having 20 faces
imaginary number	мнимое число	any complex number of the form ib, where i = $\sqrt{-1}$
improper fraction	неправильная дробь	a fraction in which the numerator has a greater absolute value or degree than the denominator
index	показатель	a number or variable placed as a superscript to the left of a radical sign indicating by its value the root to be extracted
infinity	бесконечность	the concept of a value greater than any finite numerical value
integer	целое число	any rational number that can be expressed as the sum or difference of a finite number of units, being a member of the set ...–3, –2, –1, 0, 1, 2, 3...
integral	интеграл	the limit of an increasingly large number of increasingly smaller quantities, related to the function that is being integrated (the integrand)
intersection	пересечение	a point or set of points common to two or more geometric configurations
irrational number	иррациональное число	any real number that cannot be expressed as the ratio of two integers, such as π
isosceles	равнобедренный	(of a triangle) having two sides of equal length; (of a trapezium) having the two nonparallel sides of equal length
locus	геометрическое место точек	a set of points whose location satisfies or is determined by one or more specified conditions
logarithm	логарифм	the exponent indicating the power to which a fixed number, the base, must be raised to obtain a given number or variable
lowest common denominator	наименьший общий знаменатель	the smallest integer or polynomial that is exactly divisible by each denominator of a set of fractions
lowest common multiple	наименьшее общее кратное	the smallest number or quantity that is exactly divisible by each member of a set of numbers or quantities
matrix	матрица	a substance, situation, or environment in which something has its origin, takes form, or is enclosed

Term	*Translation*	*Definition*
mean	среднее	the second and third terms of a proportion, as b and c in a/b = c/d
median	медиана	a straight line joining one vertex of a triangle to the midpoint of the opposite side
minus	минус	reduced by the subtraction of
mode	мода	that one of a range of values that has the highest frequency as determined statistically
multiplication	умножение	an arithmetical operation, defined initially in terms of repeated addition
natural logarithm	натуральный логарифм	a logarithm to the base “e”
natural number	натуральное число	any of the numbers 1, 2, 3, 4,… that can be used to count the members of a set; the non-negative integers
nonagon	девятиугольник	a polygon having nine sides
number	число	a concept of quantity that is or can be derived from a single unit, the sum of a collection of units, or zero
numerator	числитель	the dividend of a fraction
oblong	продолговатый	having an elongated, esp. rectangular, shape
obtuse angle	тупой угол	an angle greater than 90° but less than 180°
octagon	восьмиугольник	a polygon having eight sides
octahedron	октаэдр	a solid figure having eight plane faces
odd	нечётный	(of a number) not divisible by two
open set	открытое множество	a set which is not a closed set
operation	операция	any procedure, such as addition, multiplication, involution, or differentiation, in which one or more numbers or quantities are operated upon according to specific rules
operator	оператор	any symbol, term, letter, etc, used to indicate or express a specific operation or process
ordinal number	порядковое числительное	a number denoting relative position in a sequence
origin	начало координат	the point of intersection of coordinate axes or planes
parabola	парабола	a conic section formed by the intersection of a cone by a plane parallel to its side
parallel	параллельный	separated by an equal distance at every point; never touching or intersecting
parallelogram	параллелограмм	a quadrilateral whose opposite sides are parallel and equal in length

Term	*Translation*	*Definition*
pentagon	пятиугольник	a polygon having five sides
percentage	процент	proportion or rate per hundred parts
pi	число пи	a transcendental number, fundamental to mathematics, that is the ratio of the circumference of a circle to its diameter; approximate value: 3.141 592...; symbol: π
plus	плюс	increased by the addition of
polygon	многоугольник	a closed plane figure bounded by three or more straight sides that meet in pairs in the same number of vertices, and do not intersect other than at these vertices
polyhedron	многогранник	a solid figure consisting of four or more plane faces (all polygons), pairs of which meet along an edge, three or more edges meeting at a vertex
polynomial	многочлен	a mathematical expression consisting of a sum of terms each of which is the product of a constant and one or more variables raised to a positive or zero integral power
power	степень	the value of a number or quantity raised to some exponent
prime number	простое число	an integer that cannot be factorized into other integers but is only divisible by itself or 1
prism	призма	a transparent polygonal solid, often having triangular ends and rectangular sides, for dispersing light into a spectrum or for reflecting and deviating light; they are used in spectroscopes, binoculars, periscopes, etc
probability	вероятность	a measure or estimate of the degree of confidence one may have in the occurrence of an event, measured on a scale from zero (impossibility) to one (certainty); it may be defined as the proportion of favourable outcomes to the total number of possibilities if these are indifferent (mathematical probability), or the proportion observed in a sample (empirical probability), or the limit of this as the sample size tends to infinity (relative frequency), or by more subjective criteria (subjective probability)
product	произведение	the result of the multiplication of two or more numbers, quantities, etc
proof	доказательство	a sequence of steps or statements that establishes the truth of a proposition

Term	*Translation*	*Definition*
proper fraction	правильная дробь	a fraction in which the numerator has a lower absolute value than the denominator
quadrant	квадрант	a quarter of the circumference of a circle
quadratic equation	квадратное уравнение	an equation containing one or more terms in which the variable is raised to the power of two, but no terms in which it is raised to a higher power
quadrilateral	четырёхугольник	a polygon having four sides
quotient	частное	the result of the division of one number or quantity by another
radian	радиан	an SI unit of plane angle; the angle between two radii of a circle that cut off on the circumference an arc equal in length to the radius; 1 radian is equivalent to 57.296 degrees and π/2 radians equals a right angle
radius	радиус	a straight line joining the centre of a circle or sphere to any point on the circumference or surface
ratio	отношение	a quotient of two numbers or quantities
rational number	рациональное число	any real number of the form a/b, where “a” and “b” are integers and “b” is not zero
real number	действительное число	a number expressible as a limit of rational numbers
reciprocal	обратное число	a number or quantity that when multiplied by a given number or quantity gives a product of one
rectangle	прямоугольник	a parallelogram having four right angles
recurring decimal	периодическая десятичная дробь	a rational number that contains a pattern of digits repeated indefinitely after the decimal point
reflex angle	отражённый угол	an angle greater than 180° and less than 360°
remainder	остаток	the amount left over when one quantity cannot be exactly divided by another
rhombus	ромб	an oblique-angled parallelogram having four equal sides
right angle	прямой угол	the angle between two radii of a circle that cut off on the circumference an arc equal in length to one quarter of the circumference; an angle of 90° or π/2 radians
right-angled triangle	прямоугольный треугольник	a triangle one angle of which is a right angle
scalene	разносторонний	(of a triangle) having all sides of unequal length

Term	*Translation*	*Definition*
secant	секанс	a trigonometric function that in a right-angled triangle is the ratio of the length of the hypotenuse to that of the adjacent side; the reciprocal of cosine
sector	сектор	either portion of a circle included between two radii and an arc
semicircle	полуокружность	one half of a circle
set	множество	a collection of numbers, objects, etc, that is treated as an entity
simultaneous equations	система уравнений	a set of equations that are all satisfied by the same values of the variables
sine	синус	a trigonometric function that in a right-angled triangle is the ratio of the length of the opposite side to that of the hypotenuse
solid	тело	a closed surface in three-dimensional space
sphere	сфера	the solid figure bounded by this surface or the space enclosed by it
square	квадрат	a plane geometric figure having four equal sides and four right angles
square root	квадратный корень	a number or quantity that when multiplied by itself gives a given number or quantity
subset	подмножество	a set within a larger set
subtraction	вычитание	a mathematical operation in which the difference between two numbers or quantities is calculated
sum	сумма	the result of the addition of numbers, quantities, objects, etc
tangent	тангенс	a geometric line, curve, plane, or curved surface that touches another curve or surface at one point but does not intersect it; a trigonometric function that in a right-angled triangle is the ratio of the length of the opposite side to that of the adjacent side; the ratio of sine to cosine
tetrahedron	тетраэдр	a solid figure having four plane faces
torus	тор	a ring-shaped surface generated by rotating a circle about a coplanar line that does not intersect the circle
trapezium	трапеция	a quadrilateral having two parallel sides of unequal length
triangle	треугольник	a three-sided polygon
union	объединение	a set containing all members of two given sets

Term	*Translation*	*Definition*
universal set	универсальное множество	the set of all objects or elements considered in a given problem
value	значение	a particular magnitude, number, or amount
variable	переменная	an expression that can be assigned any of a set of values
vector	вектор	a variable quantity, such as force, that has magnitude and direction and can be resolved into components that are odd functions of the coordinates
Venn diagram	диаграмма Венна	a diagram in which mathematical sets or terms of a categorial statement are represented by overlapping circles within a boundary representing the universal set, so that all possible combinations of the relevant properties are represented by the various distinct areas in the diagram
volume	объём	the magnitude of the three-dimensional space enclosed within or occupied by an object, geometric solid, etc
vulgar fraction	обыкновенная дробь	a fraction in which the numerator and denominator are both integers expressed as a ratio rather than a decimal
x-axis	ось абсцисс	the horizontal, or more nearly horizontal, axis in a plane Cartesian coordinate system, along which the abscissa is measured
y-axis	ось ординат	the vertical, or more nearly vertical, axis in a plane Cartesian coordinate system, along which the ordinate is measured
z-axis	ось аппликат	in a three-dimensional Cartesian coordinate system, the axis that is perpendicular to the x-axis and the y-axis and that is used to measure or plot the values of the applicate

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