"Mathematics is the queen of sciences and number theory is the queen of mathematics."

(Carl Friedrich Gauss)

Module 2. Number Theory and Algebra

Unit 6. Arithmetic

Activity 44. Watch the short film entitled "Alternative Math" to choose the best answer to the questions. What could be meant by the phrase "alternative math"? Watch the video again and make a note of all the mathematical words used in the film. Explain the meaning behind the title of the film.

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Watch the video

1. What was Danny's mistake on the math test?
A. He didn't complete the test
B. He wrote the numbers next to each other instead of adding them
C. He refused to answer the question
D. He added the numbers incorrectly and got five

2. Why did Danny's parents come to the school?
A. To thank Mrs. Wells for helping their son
B. To discuss Danny's excellent performance
C. To complain about Mrs. Wells saying Danny's answer was wrong
D. To ask for extra homework for Danny

3. What did the principal suggest Mrs. Wells should do about the situation?
A. Give Danny a better grade
B. Apologize to the parents
C. Call the police
D. Transfer Danny to another class

4. What can we infer about the principal's attitude towards education?
A. He believes teachers should maintain academic standards
B. He thinks parents' opinions are more important than correct answers
C. He supports Mrs. Wells completely
D. He wants to improve the math curriculum

5. What happened to Mrs. Wells at the end of the story?
A. She received a promotion
B. She was given more students to teach
C. She lost her job at the school
D. She was asked to teach a different subject

Activity 45. Notice the two different ways of pronouncing the word "arithmetic" depending on its part of speech. Practise reading aloud the sentences.

aRITHmetic (noun) arithMETic (adjective) = arithmetical

1. **Arithmetic** is the branch of mathematics that deals with the study of numbers and the basic operations of addition, subtraction, multiplication, and division.
2. Children often start learning **arithmetic** in elementary school to build a strong foundation in mathematical concepts.
3. Mastery of **arithmetic** is essential for success in more advanced branches of mathematics.
4. Mental **arithmetic** involves performing calculations in one's mind without the use of external aids.
5. The **arithmetic** calculations were straightforward, requiring only basic mathematical operations.
6. The **arithmetic** exercise book contained a variety of problems to test the students' computational skills.
7. The **arithmetic** lesson covered fundamental mathematical principles, preparing students for more advanced coursework.
8. The **arithmetic** problems became more complex as the students progressed through the curriculum.
9. The **arithmetic** skills of the students were evident as they effortlessly solved the mathematical problems.
10. The teacher focused on teaching **arithmetic** to enhance the students' numerical abilities.

Activity 46. Role-play the conversation between Alice, the White Queen, and the Red Queen from Lewis Carroll's "Through the Looking-Glass". Find the names of arithmetical operations.

"Manners are not taught in lessons," said Alice. "Lessons teach you to do sums, and things of that sort."

"And you do addition?" the White Queen asked. "What's one and one and one and one and one and one and one and one and one and one?"

"I don't know," said Alice. "I lost count."

"She can't do addition," the Red Queen interrupted. "Can you do subtraction? Take nine from eight."

"Nine from eight I can't, you know," Alice replied very readily: "but —"

"She can't do subtraction," said the White Queen. "Can you do division? Divide a loaf by a knife — what's the answer to that?"

"I suppose —" Alice was beginning, but the Red Queen answered for her. "Bread-and-butter, of course. Try another subtraction sum. Take a bone from a dog: what remains?"

Alice considered. "The bone wouldn't remain, of course, if I took it — and the dog wouldn't remain; it would come to bite me — and I'm sure I shouldn't remain!"

"Then you think nothing would remain?" said the Red Queen.

"I think that's the answer."

"Wrong, as usual," said the Red Queen: "the dog's temper would remain."

"But I don't see how —"

"Why, look here!" the Red Queen cried. "The dog would lose its temper, wouldn't it?"

"Perhaps it would," Alice replied cautiously.

"Then if the dog went away, its temper would remain!" the Queen exclaimed triumphantly.

Alice said, as gravely as she could, "They might go different ways." But she couldn't help thinking to herself, "What dreadful nonsense we are talking!"

"She can't do sums a bit!" the Queens said together, with great emphasis.

"Can *you* do sums?" Alice said, turning suddenly on the White Queen, for she didn't like being found fault with so much.

The Queen gasped and shut her eyes. "I can do addition, if you give me time — but I can't do subtraction, under any circumstances!"

(from "Through the Looking-Glass, and What Alice Found There," by Lewis Carroll, 1871)

Activity 47. Read the dialogue between Alice and the Mock Turtle from Lewis Carroll's "Alice in Wonderland". Explain the puns in bold.

"I only took the regular course."

"What was that?" inquired Alice.

"**Reeling** and **Writhing**, of course, to begin with," the Mock Turtle replied; "and then the different branches of Arithmetic — **Ambition**, **Distraction**, **Uglification**, and **Derision**."

"I never heard of '**Uglification**,'" Alice ventured to say. "What is it?"

(from "Alice's Adventures in Wonderland," by Lewis Carroll, 1865)

Figure 2. The Gryphon and the Mock Turtle (by Sir John Tenniel)

Activity 48. In pairs, discuss the questions.

1. What are the three Rs?
2. What role do the three Rs play in education?
3. What are the four basic operations of elementary arithmetic?
4. Is mental arithmetic (mental calculation) an essential skill? Why?
5. Why is it integral to have the multiplication table (times table) memorized?

Activity 49. Read the article to discriminate between arithmetic and number theory.

The branch of mathematics concerned with computations using numbers is called arithmetic. This can involve a number of specific topics — the study of operations on numbers, such as addition, multiplication, subtraction, division, and square roots, needed to solve numerical problems; the methods needed to change numbers from one form to another (such as the conversion of fractions to decimals and vice versa); or the abstract study of the number systems, number theory, and general operations on sets as defined by group theory and modular arithmetic, for instance.

The word arithmetic comes from the Greek word “arithmetiké”, constructed from “arithmós” meaning “number” and “techné” meaning “science.” In the time of ancient Greece, the term “arithmetic” referred only to the theoretical work about numbers, with the word “logistic” used to describe the practical everyday computations used in business. Today the term “arithmetic” is used in both contexts.

The study of the arithmetic properties of numbers is called number theory. The fact that many simple statements about numbers can be extraordinarily difficult to prove, if at all possible, makes this topic an alluring and stimulating subject for mathematicians. (Goldbach’s conjecture, for instance, remains unsolved.)

Elementary number theory is the study of those topics in number theory that utilize only the basic techniques of arithmetic and high-school mathematics in their solutions. For example, the classification of the Pythagorean triples would be considered a problem in elementary number theory, as would the solution of many Diophantine equations. (The use of the word “elementary” here by no means implies that the level of mathematical sophistication used is elementary.) Analytic number theory incorporates the notion of limit in the study of numbers, and algebraic number theory extends the study of number theory to a general study of algebraic numbers and new number systems that include solutions to otherwise unsolvable algebraic equations.

(from Encyclopaedia Britannica)

Activity 50. Study the tables.

Table 17

Addition				
1 + 2 = 3		one plus two equals three one and two is/make/give three		
addend summand augend	**plus sign**	**addend summand**	**equal(s) sign equality sign**	**sum**
1	+	2	=	3

Table 18

Subtraction				
3 – 2 = 1		three minus two equals one two from three is one		
minuend	**minus sign**	**subtrahend**	**equal(s) sign equality sign**	**difference**
3	–	2	=	1

Table 19

Multiplication				
2 × 3 = 6 2 · 3 = 6		two multiplied by three equals six two times three is six		
factor multiplier	**multiplication sign**	**factor multiplicand**	**equal(s) sign equality sign**	**product**
2	x (times sign) · (raised dot)	3	=	6

Table 20

Division					
6 ÷ 3 = 2 7 ÷ 3 = 2 (1)	**6/3 = 2 7/3 = 2 (1)**	six divided by three equals two seven divided by three equals two, remainder one			
dividend	**division sign**	**divisor**	**equal(s) sign equality sign**	**quotient**	**remainder**
6	÷ (obelus) / (slash)	3	=	2	
7	÷ (obelus) / (slash)	3	=	2	1

Table 21

Exponentiation (Power)			
$3^2 = 9$ three squared equals nine three (raised) to the power of two equals nine three (raised) to the second (power) equals nine the second power of three is nine		**$2^3 = 8$** two cubed equals eight two (raised) to the power of three equals eight two (raised) to the third (power) equals eight the third power of two is eight	
base	**exponent** **power** **index**	**equal(s) sign** **equality sign**	**power**
3	^{2}	=	9
2	^{3}	=	8

Table 22

Extraction (Root)				
$\sqrt{4}$ **= 2** the square root of four is two $\sqrt[3]{8}$ **= 2** the cube root of eight is two $\sqrt[4]{16}$ **= 2** the fourth root of sixteen is two				
degree **index**	**radical sign** **root sign**	**radicand**	**equal(s) sign** **equality sign**	**root**
^{2}	√	4	=	2
^{3}	√	8	=	2
^{4}	√	16	=	2

Table 23

Equality			
$=$	is equal to	$a = b$	A is equal to B
Inequation (Inequality)			
$\neq$	is not equal to	$a \neq b$	A is not equal to B
Inequality			
$>$	is greater than	$a > b$	A is greater than B
$<$	is less than	$a < b$	A is less than B
$\geq$	is greater than or equal to	$a \geq b$	A is greater than or equal to B
$\leq$	is less than or equal to	$a \leq b$	A is less than or equal to B

Activity 51. In pairs, calculate the expressions and read them out loud. Give the names of the elements.

Addition	Subtraction
1) 9 + 3 = 2) 27 + 436 = 3) 4 + 36 + 19 = 4) 236 + 782 = 5) 5,345 + 655 = 6) 2 + 1 + 38 + 3 + 6 = 7) 4,447 + 7,478 + 676 = 8) 32,812 + 65,034 + 54,323 = 9) –6 + 3 = 10) –72 + (–73) = 11) 8 + (–6) + (–9) + 5 + 1 = 12) (–31 + 12) + (3 + (–16)) = 13) –24 + (–3) + 24 + (–5) + 5 =	14) 148 – 87 = 15) 343 – 269 = 16) 10,435 – 10,218 = 17) 5,231 – 5,177 = 18) 7,800 – 5,725 = 19) –7 – 6 = 20) –7 – (–6) = 21) 82 – (–109) = 22) 0 – 15 = 23) –60 – 50 – 40 =
Multiplication	**Division**
24) 49 × 9 = 25) 5 × 7 × 6 = 26) 72 × 10,000 = 27) 110 × 440 = 28) 157 · 59 = 29) 3,723 · 46 = 30) 5,624 · 281 = 31) 502 · 459 = 32) 8 × 0 = 33) 7 × 1 = 34) –10 × 7 = 35) –4(–73) = 36) –4(2) (–6) = 37) –9(–3) (–1) (–2) = 38) –20,000(1,300) =	39) 72 ÷ 4 = 40) 595 ÷ 35 = 41) 1,443 ÷ 39 = 42) 20,876 ÷ 68 = 43) 1,269 ÷ 54 = 44) 405 ÷ 21 = 45) 0 ÷ 10 = 46) 5,347 ÷ 127 = 47) 1,482,000 ÷ 3,900 = 48) –5 ÷ 1 = 49) 0 ÷ (–6) = 50) –18 ÷ (–18) =

Activity 52. In pairs, calculate the expressions and read them out loud.

Power	Root
1) 1^9	16) $\sqrt{12}$
2) 2^5	17) $\sqrt{121}$
3) 2^8	18) $\sqrt{144}$
4) 3^4	19) $\sqrt{16}$
5) 3^7	20) $\sqrt[3]{1}$
6) 4^5	21) $\sqrt[3]{125}$
7) 5^2	22) $\sqrt[3]{216}$
8) 5^3	23) $\sqrt[3]{27}$
9) 6^1	24) $\sqrt[4]{256}$
10) 6^3	25) $\sqrt[4]{81}$
11) 7^2	26) $\sqrt{49}$
12) 7^3	27) $\sqrt{64}$
13) 8^2	28) $\sqrt{81}$
14) 9^1	29) $\sqrt{9}$
15) 9^2	

Activity 53. In pairs, put an inequality sign and read the statements out loud.

1) 9 __ 7
2) 301 __ 310
3) 0 __ –7
4) –20 __ –19
5) –8 __ –9
6) –213 __ 123
7) –5 __ 0
8) 5.68 __ 5.75
9) 106.8199 __ 106.82
10) –78.23 __ –78.303
11) –555.098 __ –555.0991

Activity 54. Create ten expressions using addition, subtraction, multiplication, division, exponentiation, and extraction as well as five inequalities. Read them aloud for your partner to write down symbolically. Check if the expressions are correct. Swap roles.

Activity 55. Complete the divisibility rules.

1. All numbers are divisible by __.
2. A number is divisible by __ only if its final digit is 0, 2, 4, 6 or 8.
3. A number is divisible by __ if its final two digits represent a two-digit number that can be divided by 2 twice.
4. A number is divisible by __ only if its final digit is 0 or 5.
5. A number is divisible by __ only if it is an even number whose digits sum to a multiple of 3.
6. A number is divisible by __ if its final three digits represent a three-digit number that can be divided by 2 three times.
7. A number is divisible by __ only if its final digit is a zero.
8. A number is divisible by __ only if it is divisible by both 3 and 4.

Activity 56. Decipher the mnemonics used to memorize the order of precedence.

1. BEDMAS
2. PEMDAS

Activity 57. Complete the table with the properties of operations.

associative property / commutative property / distributive property	
Property	**Operations**
(1) ___________	a + (b + c) = (a + b) + c a × (b × c) = (a × b) × c
(2) ___________	a + b = b + a a × b = b × a
(3) ___________	a × (b + c) = a × b + a × c

Activity 58. Design an informative and illustrated classroom poster on the basic operations of elementary arithmetic, their order of precedence, and properties.

Unit 7. Numeral Systems and Base Systems

Activity 59. Complete the table with the Hindu-Arabic equivalents of the Roman numerals.

Roman	I	V	X	L	C	D	M
Hindu Arabic							

Activity 60. Match the corresponding numerals in the two columns.

(1) 2,000
(2) 4
(3) 6
(4) 13
(5) 17
(6) 27
(7) 29
(8) 44
(9) 60
(10) 90
(11) 173
(12) 360
(13) 444
(14) 600
(15) 800
(16) 999
(17) 1170
(18) 1250

(a) CCCLX
(b) CDXLIV
(c) CLXXIII
(d) CMXCIX
(e) DC
(f) DCCC
(g) IV
(h) LX
(i) MCCL
(j) MCLXX
(k) MM
(l) VI
(m) XC
(n) XIII
(o) XLIV
(p) XVII
(q) XXIX
(r) XXVII

Activity 61. Watch the video "A Brief History of Numerical Systems" to choose the best answer to the questions. What systems of numeration are you aware of? Expand on the revolutionary nature of the place-value system and positional notation.

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1. What was the main problem with early number systems like Greek, Hebrew, and Egyptian numerals?
A. They could only represent numbers up to one hundred
B. They required repeating symbols many times and creating new symbols for larger numbers
C. They were too difficult for merchants to understand
D. They could not represent the number zero at all

2. How does positional notation improve upon earlier number systems?
A. It uses the same symbols in different positions to represent different values
B. It requires fewer people to learn how to count properly
C. It eliminates the need for mathematical calculations
D. It works only with the number ten and its multiples

3. Why was the invention of zero considered a key breakthrough?
A. It allowed people to count backwards for the first time
B. It made mathematics easier to teach in schools
C. It prevented confusion between numbers like 63 and 603
D. It replaced all other symbols in the number system

4. Why do most number systems use base 10?
A. Because it is the most mathematically efficient system
B. Because ancient mathematicians proved it was superior
C. Because humans have ten fingers
D. Because it was required by early governments

5. What advantage does a base 12 system have over base 10?
A. It uses fewer symbols to write large numbers
B. It can be divided evenly by more numbers
C. It is easier for children to learn
D. It works better with modern computers

Activity 62. Read the article to outline the system of Roman numerals and that of Hindu-Arabic numerals.

More than 5,000 years ago an Egyptian ruler recorded, perhaps with a bit of exaggeration, the capture of 120,000 prisoners, 400,000 oxen, and 1,422,000 goats. This event was inscribed on a ceremonial mace which is now in a museum in Oxford, England.

The ancient Egyptians developed the art of counting to a high degree, but their system of numeration was very crude. For example, the number 1,000 was symbolized by a picture of a lotus flower and the number 2,000 was symbolized by a picture of two lotus flowers growing out of a bush. Although these symbols, called hieroglyphics, permitted the Egyptians to write large numbers, the numeration system was clumsy and awkward to work with. The number 999, for instance, required 27 individual marks.

In our system of numeration, we use ten symbols called digits — 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 — and combinations of these symbols. Our system of numeration is called decimal, or base-ten, system. There is little doubt that our ten fingers influenced the development of a numeration system based on ten digits.

The ancient Hindus are credited with discovering the decimal system of numeration we use today. This system was translated into Arabic prior to its introduction into Europe by travelling merchants around the 13th century. Hence it is also known as the Hindu-Arabic system. Adoption of the Hindu-Arabic system met resistance due to the widespread use of the Roman numeral system during this period.

Based on a simple tally system similar to the one used by the ancient Egyptians, merchants of the Roman empire of about 500 B.C.E. used letter symbols for powers of 10 and for the intermediate values of 5, and simply grouped symbols together to represent all other quantities.

The expression CLXXIII, for instance, represented the number 100 + 50 + 10 + 10 + 1 + 1 + 1 = 173. Although the order of the symbols was not important, it became the convention to list symbols from largest to smallest, left to right.

Initially the symbols D and M were not part of the Roman system. The number 1,000 was written (I), and further applications of round brackets allowed for the expression of even greater quantities. For instance, ((I)) represented 10,000, and (((I))) represented 100,000. Stonemasons introduced the symbols D and M to simplify their work.

The Romans also introduced other ornamentations to increase the value of a numerical symbol. For instance, vertical bars were used to represent a 100-fold increase, and a bar placed above the symbol represented a 1,000-fold increase.

There was no symbol for zero in the Roman system. To avoid the four-fold repetition of symbols (as in the expression CCCCXXXXIIIII for 444), a subtractive principle was introduced in the 13th century:

The placement of a small value immediately to the left of a higher value indicates that that small value is to be subtracted from the higher value.

Thus, 4 could be written as IV, 90 as XC, and 444 as CDXLIV. The subtractive principle was subject to two rules:

1. *The symbols V, L, and D cannot be used as the numbers to be subtracted.*
2. *Only one symbol I, X, or C can be placed before a higher number symbol.*

Thus, for example, it was not permissible to write IIX for eight. Although not a proper place-value system, with the subtractive principle in use, the order of the symbols used was now important.

Performing operations of basic arithmetic with Roman numerals is very awkward. For example, it is not immediate what the solution to the following addition problem should be:

XLIV + XVII + XXIX

That European merchants were comfortable working with the Roman numeral system for well over a millennium suggests that scholars did not use the numeral system to perform calculations, only to record the results. (Arithmetic was performed on a counting board such as an abacus.)

The system of Roman numerals remained popular in Western Europe until the 17^{th} century. Although the system was eventually replaced by the Hindu-Arabic numeral system we use today, it still remains a tradition to use Roman numerals for numbering introductory pages in books, for instance.

Using a base-10 place-value system, numbers in the Hindu-Arabic system are expressed via combinations of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, organized so as to represent groupings of powers of 10. For instance, the number 574 represents the five groups of 100, seven groups of 10, and four single units.

This numerical system originated from India around 600 C.E., almost in the exact same form as we use it today. The system was transmitted to the Arabs two centuries later as they worked to translate the Sanskrit works on astronomy into Arabic. The Arab mathematician Muhammad ibn Musa al-Khwarizmi (ca. 800) wrote an influential treatise describing the Hindu numeral system and used it in his famous book “Calculation by Restoration and Reduction”, from whose title the modern word “algebra” is derived. As Western scholars began translating the Arabic texts into Latin, word of the efficient numeration system spread across Western Europe. The Italian scholar Fibonacci (ca. 1170–1250) avidly promoted their use. By the end of the 17^{th} century, the Hindu-Arabic numeral system completely replaced the cumbersome system of Roman numerals that were the standard in Europe for over 1,500 years.

Other numeration systems were developed in early cultures and societies. Two of the most common were the base-five system, related to the number of fingers on one hand and the base-twenty system, related to the number of fingers and toes.

In some languages the word for “five” is the same as the word for “hand”, and the word for “ten” is the same as the word for “two hands”. In the English language the word “digit” is a synonym for the word “finger” — that is, ten digits, ten fingers.

Still another early system of numeration was a base-sixty system developed by the Mesopotamians and used for centuries. These ancient people divided the years into 360 days (6 × 60); today we still divide the hour into 60 minutes and the minute to 60 seconds. Numeration systems of current interest include a binary, or base-two, system used in electronic computers and a base-twelve, or duodecimal system.

(from Elementary Algebra)

Activity 63. Reorder the sentences to make a text on numeral systems.

a. A place-value system, such as the Arabic numeral system, has clear advantages in economy of symbolism and in efficiency of computation.
b. For example, in 333, the 3 on the right means three, but the 3 in the middle means three tens and the 3 on the left means three hundreds.
c. In modern Roman numerals (where I, V, X, L, C, D, and M are 1, 5, 10, 50, 100, 500, and 1,000, respectively), on the other hand, CCC means 300 — each C stands for one hundred, and the relative position of the C's is of no importance.
d. In this system the position a symbol occupies helps determine the value of the symbol.
e. Only the 10 symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 and the decimal point are needed to write numbers of any size.
f. The numeration system (or system for writing number symbols) widely used throughout the world today is a place-value system based on the number 10 and usually called the Arabic, or Hindu-Arabic, numeration system.
g. There are some rules regarding the order of symbols in the Roman numeral system, however (for example, IX means 9, while XI means 11), though generally position is not as important as in place-value systems.

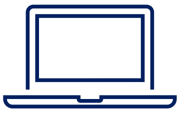
Activity 64. Develop a system of criteria for comparing numeral systems. Using it, contrast the Hindu-Arabic numerical system with the Roman system of numeration. Argue for the widespread use of the former in today's world.

Activity 65. Draw a mind map showing the application of several base systems of your choice.

Table 24. Base Systems

System	Base Value	System	Base Value	System	Base Value
		undenary	11		
binary	2	duodecimal	12	vigesimal	20
ternary	3				
quaternary	4				
quinary	5				
senary	6	hexadecimal	16	sexagesimal	60
septenary	7				
octal	8				
nonary	9				
decimal	10				

Unit 8. Number Sets



Activity 66. Complete the sentences with the plural form of the words in brackets.

1. __________ (hyperbola) take different shapes in analytic geometry.
2. __________ (matrix) simplify complex calculations in linear algebra.
3. __________ (polyhedron) include figures like cubes and prisms.
4. __________ (rhombus) are a type of parallelograms.
5. __________ (torus) exhibit unique topological properties.
6. Calculate the areas of these __________ (trapezium).
7. Earth's rotation involves multiple __________ (axis).
8. Ensure proper use of __________ (parenthesis) in mathematical expressions.
9. Mathematicians rely on many __________ (lemma) to prove theorems.
10. Natural __________ (phenomenon) are studied in physics.
11. Postgraduate students defend their __________ (thesis).
12. Roll the __________ (die) to simulate different outcomes.
13. The __________ (analysis) of the data reveal interesting patterns.
14. The __________ (locus) of points form intricate patterns.
15. The book explores captivating __________ (series) of mathematical ideas.
16. The converging lines meet at various __________ (vertex).
17. The event marked significant changes over several __________ (millennium).
18. Mathematical __________ (proof) confirm the validity of theorems.
19. The mountain range has several towering __________ (apex).
20. The scientists conducted lots of __________ (research) on the topic.
21. The scientists derived different __________ (formula) for their experiments.
22. The scientists tested several __________ (hypothesis) to explain the phenomenon.
23. The success __________ (criterion) for the project are diverse.
24. Various circles have distinct __________ (radius).

Activity 67. Watch the video "Making Sense of Irrational Numbers" to choose the best answer to the questions. Describe how Hippasus' discovery revolutionized mathematics.

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1. What was Hippasus' "crime" according to the Greek myth?
A. He murdered his guests at a religious ceremony
B. He discovered irrational numbers through mathematical proof
C. He challenged Pythagoras to a public debate
D. He stole sacred mathematical texts from the gods

2. What did the Pythagorean mathematicians believe about numbers?
A. Numbers were useful tools for counting objects
B. Numbers were the building blocks of the universe and could all be expressed as ratios
C. Numbers were created by the gods and shouldn't be studied
D. Numbers were less important than geometry and shapes

3. How did Hippasus prove that the square root of 2 could not be expressed as a ratio?
A. He used a proof by contradiction to show that assuming it was rational led to an impossible situation
B. He calculated the decimal expansion and showed it never ended
C. He measured the diagonal of a square and found it was impossible
D. He asked other mathematicians and they all agreed with him

4. What is the key point about irrational numbers like the square root of 2 and pi?
A. They are mistakes in mathematics that should be avoided
B. They can eventually be expressed as ratios if we try hard enough
C. Decimals and ratios are just ways to express numbers, but these numbers have exact values
D. They are only theoretical and cannot be represented in any way

5. Why is forming right triangles on a number line mentioned?
A. To show that irrational numbers can be precisely plotted even though they can't be expressed as ratios
B. To prove that the square root of 2 is actually a rational number
C. To demonstrate that geometry is more important than algebra
D. To explain how the Pythagoreans originally discovered these numbers

Activity 68. In pairs, discuss the questions.

1. What do the letters N, Z, Q, R, C mean to a mathematician?
2. Define N, Z, Q, R, C in terms of subsets and supersets.
3. Represent the relations between N, Z, Q, R, C symbolically.

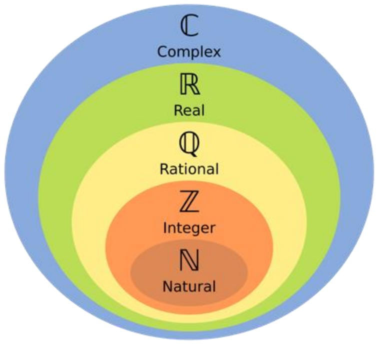

Figure 3. Number Systems

Activity 69. Read the article to complete the table based on the text.

The development of different types of numbers can be seen as motivated by the need for solving different types of equations. For example, the counting numbers (that is, the natural numbers N) suffice for solving any equation of the type x + 2 = 5, for instance, but not an equation of the type x + 5 = 2. (There is no solution to this equation within the set of counting numbers.) This motivates the introduction of negative numbers and the construction of the integers Z (from the German word “Zahlen” for “numbers”).

Working solely in the realm of the whole numbers, a number is said to be even if it is divisible by 2, and odd if it leaves a remainder of 1 when divided by 2. For example, 18 is divisible by 2 and so is even, and 23 leaves a remainder of 1 and so is odd. As the study of evenness and oddness shows, the number 0 is even. Two integers that are either both even or both odd are said to have the same parity. For instance, 17 and 53 have the same parity (both are odd), and 9 and 14 have opposite parity. Sometimes the term “parity” is used in a more general setting as to mean “being in one of two possible states” (either positive or negative).

A whole number possessing just two positive factors is called a prime number, or simply a prime. For example, 7 has only two positive factors, namely 1 and 7, and so is prime. The number 24 has eight positive factors and so is not prime, and the number 1 has only one factor and is not prime. The term composite is used to describe numbers greater than 1 that are not prime, that is positive whole numbers with more than two positive factors. (Medieval mathematician Fibonacci (1170–1250) called prime numbers “incomposite.”) It is vital that the number 1 be considered neither prime nor composite for the fundamental theorem of arithmetic to hold true.

But the set of integers is not always sufficient for solving equations of the type 5x = 3, for instance. Desiring solutions to equations of this type leads to the construction of fractions and the set of all rational numbers Q (for “quotient”). A rational number is any number that can be written in the form a/b, where “a” and “b” represent integers and $b \neq 0$. The set of rational numbers is the set of all terminating and all repeating decimals.

Unfortunately, again, not all equations can be solved within the rational system. For example, the equation $x^2 - 2 = 0$ has no rational solution. Extending the set of rational numbers to include solutions to equations of this type introduces irrational numbers and the construction of the real number system R.

An irrational number is a nonterminating, nonrepeating decimal. An irrational number cannot be expressed as a fraction with an integer numerator and a nonzero integer denominator. Two subsets of irrationals are algebraic and transcendental numbers. A number is called algebraic if it is the root of a polynomial with integer coefficients. For example, (1/2) $(5 + \sqrt{13})$ is algebraic since it is a solution to the equation $x^2 - 5x + 3 = 0$. Numbers that are not algebraic are called transcendental.

It is extraordinarily difficult to define precisely what is meant by a real number. Many standard texts in mathematics define a real number to be any rational number or any irrational number.

A real number “x” is said to be positive if it is greater than zero, that is, if $x > 0$. A real number less than zero is called negative. An unspecified real number that is positive or possibly zero is called nonnegative. One that is negative or possibly zero is called nonpositive. Zero is the only real number that is neither positive nor negative.

Yet the system of real numbers also does not suffice for solving all equations. With the introduction of a single additional number, denoted “i”, to represent an “imaginary” solution to the equation $x^2 + 1 = 0$, the complex numbers C are born. The number “i” is usually regarded as the square root of negative one: $i = \sqrt{-1}$. (One must be careful as there are, in fact, two square roots of this quantity, namely “i” and “–i”.) Surprisingly, as shown by the fundamental theorem of algebra, the introduction of this single number is all that is needed to solve any polynomial equation $a_n x^n + ... + a_1 x + a_0 = 0$. Thus, the complex numbers represent a system of numbers that is algebraically closed in the sense that the construction of no new type of number is needed to solve arithmetic equations.

On a conceptual level, the notion of “number” is intimately connected with the act of counting. Simple counting systems of ancient times used tally marks to record numbers, and over the millennia this basic numeration scheme evolved to the sophisticated place-value system we use today. (The ancient Egyptians of around 3000 B.C.E. were perhaps the first to move from the use of tally marks alone.) It was a great intellectual achievement for mankind when the notion of “number” was removed from the specific objects being counted, recognizing, for instance, that two cows, two houses, and two days all share a common property of “two-ness.” (Even today we sometimes use different words to count different types of “two.” For instance, the words “twins”, “couple”, and “pair” cannot be used interchangeably to represent two people.) This simple recognition of an abstract commonality between sets of objects was exploited by German mathematician Georg Cantor (1845–1918) who, in the late 1800s, developed a general notion of cardinality. With it, Cantor extended the notion of "number" to include counts of sets of infinite size. He established, for instance, that there are an infinite number of different types of infinity and managed to develop a meaningful system of arithmetic for his transfinite numbers.

The Irish mathematician Sir William Rowan Hamilton (1805–65) followed a different route and worked to extend the notion of "number" to represent operations on n-dimensional space. An Argand diagram shows that the complex numbers have a natural representation as points on a plane. Hamilton sought to give meaning to an arithmetic for points in three- and higher-dimensional space. Although he did not succeed in accomplishing this goal for three-dimensional space, his invention of the quaternions shows this feat can be done in four-dimensional space. (The octonions provide an arithmetic for eight-dimensional space.)

(from Elementary Algebra)

№	**Set**	**Type**	**Definition**	**Examples**
1	N	a natural number		
2		a whole number		
3		an even number		
4		an odd number		
5		a prime number		
6		a composite number		
7	Z	an integer		
8	Q	a rational number		
9		an irrational number		
10		an algebraic number		
11		a transcendental number		
12	R	a real number		
13		a positive number		
14		a negative number		
15	C	a complex number		

Activity 70. Complete the parity properties (even or odd).

1. The sum of two even numbers is ___________.
2. The sum of any number of even numbers is ___________.
3. The difference of two even numbers is ___________.
4. The sum of an even number and an odd number is ___________.
5. The difference of an even number and an odd number is ___________.
6. The sum of two odd numbers is ___________.
7. The sum of an even number of odd numbers is ___________.
8. The sum of an odd number of odd numbers is ___________.
9. The difference of two odd numbers is ___________.
10. The product of two even numbers is ___________.
11. The product of an even number and an odd number is ___________.
12. The product of two odd numbers is ___________.

Activity 71. Watch the video "A Brief History of Banned Numbers" to choose the best answer to the questions. Why do you think numbers can be banned? In your view, what numbers may have been banned? Compare your ideas to the ones in the video.

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1. What did the Pythagoreans believe about mathematics?
A. It was only useful for counting objects
B. It held the deepest secrets of the universe
C. It was less important than philosophy
D. It should be kept secret from everyone

2. Why were Hindu-Arabic numerals banned in Florence?
A. They were too difficult for merchants to learn
B. Religious leaders thought they were against God
C. Authorities worried they could be easily forged or altered
D. They made it impossible to record business transactions

3. What can be inferred about Hippasus?
A. He was rewarded for his mathematical discoveries
B. His discovery challenged the Pythagoreans' beliefs about the universe
C. He invented irrational numbers to prove his teachers wrong

D. He was the most famous mathematician in ancient Greece

4. Why might some numbers be illegal today?
A. They are too large for computers to process
B. They represent protected information like copyrights or state secrets
C. They are connected to ancient mathematical theories
D. Merchants use them to cheat their customers

5. What is the main idea of the video?
A. Mathematics has always been the most important subject in education
B. Throughout history, certain numbers have been considered dangerous or banned for various reasons
C. The Pythagoreans were wrong about their mathematical theories
D. Modern governments should stop making numbers illegal

Activity 72. Write a paragraph justifying the existence of multiple number systems.

Unit 9. Algebra

Activity 73. Complete the table with the types of algebraic expressions.

equality / equation / identity / inequality / inequation	
Expression	**Example**
(1) ___________	a = b
(2) ___________	$a \neq b$
(3) ___________	$x > y$
(4) ___________	3x + 4 = 8
(5) ___________	x + 5 + x = 2x + 5

Activity 74. Choose the best alternative.

1. Constants / Variables are letters (or symbols) that stand for numbers.
2. To perform the multiplication 3(x + 4), we use the associative / distributive property.
3. Terms such as $7x^2$ and $5x^2$, which have the same variables raised to exactly the same power, are called like / similar terms.
4. The coefficient / constant of the term 9y is 9.
5. To estimate / evaluate $y^2 + 9y – 3$ for y = –5, we substitute –5 for y and apply the order of operations rule.
6. An equality / equation is a statement indicating that two expressions are equal.
7. To solve an equation means to find all magnitudes / values of the variable that make the equation true.

Activity 75. Match the algebraic structures to the axioms they satisfy.

1. group (G) [4]
2. Abelian group [5]
3. ring (R) [7]
4. commutative ring [8]
5. integral domain [9]
6. field [10]

a. associative law of addition
b. associative law of multiplication
c. associativity
d. closure
e. commutative law for addition
f. commutative law for multiplication
g. commutativity
h. distributive laws
i. existence of a 1
j. existence of a zero
k. existence of additive inverses

l. existence of an identity
m. existence of inverses
n. existence of multiplicative inverses
o. no divisors of zero

Activity 76. Read the article. Expand on what constitutes the scope of modern algebra as a distinct branch of mathematics.

The branch of mathematics concerned with the general properties of numbers, and generalizations arising from those properties, is called algebra. Often symbols are used to represent generic numbers, thereby distinguishing the topic from the study of arithmetic. For instance, the equation $2 \times (5 + 7) = 2 \times 5 + 2 \times 7$ is a (true) arithmetical statement about a specific set of numbers, whereas the equation $x \times (y + z) = x \times y + x \times z$ is a general statement describing a property satisfied by any three numbers. It is a statement in algebra.

Much of elementary algebra consists of methods of manipulating equations to either put them in a more convenient form, or to determine (that is, solve for) permissible values of the variables that appear. For instance, rewriting $x^2 + 6x + 9 = 25$ as $(x + 3)^2 = 25$ allows an easy solution for "x": either $x + 3 = 5$, yielding $x = 2$, or $x + 3 = -5$, yielding $x = -8$.

The word "algebra" comes from the Arabic term "al-jabr" used by the great Muhammad ibn Musa al-Khwarizmi (ca. 780–850) in his writings on the topic.

In modern times the subject of algebra has been widened to include abstract algebra, group theory, and the study of alternative number systems such as modular arithmetic. Boolean algebra looks at the algebra of logical inferences, matrix algebra the arithmetic of matrix operations, and vector algebra the mechanics of vector operations and vector spaces.

An algebraic structure is any set equipped with one or more operations (usually binary operations) satisfying a list of specified rules. For example, any group, ring, field, or vector space is an algebraic structure.

Research in pure mathematics is motivated by one fundamental question: what makes mathematics work the way it does? For example, to a mathematician, the question, "What is 263 × 178 (or equivalently, 178 × 263)?" is of little interest. A far more important question would be, "Why should the answers to 263 × 178 and 178 × 263 be the same?"

The topic of abstract algebra attempts to identify the key features that make algebra and arithmetic work the way they do. For example, mathematicians have shown that the operation of addition satisfies five basic principles, and that all other results about the nature of addition follow from these.

1. *Closure: The sum of two numbers is again a number.*

2. *Associativity: For all numbers "a", "b", and "c", we have: (a + b) + c = a + (b + c).*
3. *Zero element: There is a number, denoted "0," so that: a + 0 = a = 0 + a for all numbers "a".*
4. *Inverse: For each number "a" there is another number, denoted "–a," so that: a + (–a) = 0 = (–a) + a.*
5. *Commutativity: For all numbers "a" and "b" we have: a + b = b + a.*

Having identified these five properties, mathematicians search for other mathematical systems that may satisfy the same five relations. Any fact that is known about addition will consequently hold true in the new system as well. This is a powerful approach to matters. It avoids having to re-prove theorems and facts about a new system if one can recognize it as a familiar one in disguise. For example, multiplication essentially satisfies the same five axioms as above, and so for any fact about addition, there is a corresponding fact about multiplication. The set of symmetries of a geometric figure also satisfy these five axioms, and so too all known results about addition immediately transfer to interesting statements about geometry. Any system that satisfies these basic five axioms is called an "Abelian group," or just a group if the fifth axiom fails. Group theory is the study of all the results that follow from these basic five axioms without reference to a particular mathematical system.

The study of rings and fields considers mathematical systems that permit two fundamental operations (typically called addition and multiplication). Allowing for the additional operation of scalar multiplication leads to a study of vector spaces.

The theory of algebraic structures is highly developed. The study of vector spaces as well as matrices, for example, is so extensive that the topic is regarded as a field of mathematics in its own right and is called linear algebra. As matrices are used to analyze and solve systems of simultaneous linear equations and to describe linear transformations between vector spaces, this topic of study unites geometric thinking with numerical analysis. As the set of all invertible matrices of a given size form a group, called the general linear group, techniques of abstract algebra can also be incorporated into this work.

(from Elementary Algebra)

Activity 77. Study the tables.

Table 25

Axis Variable	**British**	**American**
x	/eks/	
y	/waɪ/	
z	/zed/	/zi/

Table 26

Algebraic Expression			
2x + 1 two X plus one			
Term		**Operator**	**Term**
Coefficient **Parameter**	**Variable** **Indeterminate** **Unknown**		**Constant** **Known**
2	x	+	1

Table 27

1.	**$x(y + z) = xy + xz$**
	X times the sum of Y and Z equals XY plus XZ
2.	**$ax^2 + bx^2 + c = 0$**
	AX squared plus BX squared plus C equals zero
3.	**$x^2 + 2px^2 + p^2 = (x + p)^2$**
	X squared plus two PX squared plus P squared equals the sum of X and P squared
4.	**$a(b + c) = ab + ac$**
	A times, bracket/parenthesis, B plus C, close bracket/parenthesis, equals AB plus AC
5.	**$(2a – a)/a = 1$**
	a. bracket/parenthesis, two A minus A, close bracket/parenthesis, divided by A equals one b. two A minus A ALL/QUANTITY divided by A equals one

Activity 78. While commenting on the steps, simplify and solve the equations and inequalities.

Equations	Inequalities
1) $3x – 8 – 4x – 7x = –2 – 8$ 2) $–6t – 7t – 5t – 1 = 12 – 3$ 3) $4(d – 5) + 20 = 5 – 2d$ 4) $1 – t = 5(t – 2) + 10$ 5) $30x – 12 = 1,338$ 6) $40y – 19 = 1,381$	19) $5x – 1 \leq 29$ 20) $5 + 4x > 25$ 21) $5(2 – x) \leq 30$ 22) $2x + 3x < 200 – 5x$ 23) $5(x + 2) < 6(9 – x)$ 24) $x^2 + 4x \leq x(x – 5) – 18$ 25) $x(x + 2) < x(2 – x) + 2x^2$

7) $-7 = \frac{3}{7}r + 14$	26) $2x + 3 \leq 17$
8) $21 = \frac{2}{5}f - 19$	27) $5 - 4x > 25$
9) $10 - 2y = 8$	
10) $7 - 7x = -21$	
11) $9 + 5(r + 3) = 6 + 3(r - 2)$	
12) $2 + 3(n - 6) = 4(n + 2) - 21$	
13) $-\frac{2}{3}z + 4 = 8$	
14) $-\frac{7}{5}x + 9 = -5$	
15) $-2(9 - 3s) - (5s + 2) = -25$	
16) $4(x - 5) - 3(12 - x) = 7$	
17) $9a - 2.4 = 7a + 4.6$	
18) $4c - 1.6 = 7c + 3.2$	

Activity 79. What is Blaise Pascal famous for? Watch the video "The Mathematical Secrets of Pascal's Triangle" to choose the best answer to the questions. Describe the practical applications of Pascal's triangle.

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Figure 4. Blaise Pascal

1. What does the text suggest about Pascal's Triangle's origin?
A. Pascal was the first mathematician to discover it
B. It was independently discovered by mathematicians in different cultures
C. It was originally created in France and then spread to other countries
D. Pascal stole the idea from Indian mathematicians

2. How is each new row in Pascal's Triangle created?
A. By multiplying the numbers in the previous row by two
B. By adding consecutive pairs of numbers from the row above
C. By listing all the odd numbers in sequence
D. By squaring each number from the previous row

3. What happens when you add up all the numbers in any row of Pascal's Triangle?
A. You get a triangular number

B. You get a multiple of eleven
C. You get a power of two
D. You get a prime number

4. What is the main purpose of using Pascal's Triangle in probability problems?
A. To calculate how many different combinations are possible
B. To determine which outcome is most likely to occur
C. To predict future events based on past patterns
D. To create fractal patterns for statistical analysis

5. What is implied about Pascal's Triangle?
A. It has been completely understood and has no more secrets to reveal
B. It is only useful for solving basic mathematical problems
C. Mathematicians continue to discover new applications and patterns in it
D. It is too complex for practical use in modern mathematics

Unit 10. Development of Algebra

Activity 80. Complete the table with the types of equations.

a cubic equation / a linear equation / a quadratic equation / a quartic equation / a quintic equation	
Equation	**General Form ($a \neq 0$)**
(1) ___________	$ax + b = c$
(2) ___________	$ax^2 + b + c = 0$
(3) ___________	$ax^3 + bx^2 + cx + d = 0$
(4) ___________	$ax^4 + bx^3 + cx^2 + dx + e = 0$
(5) ___________	$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$

Activity 81. Do the quiz on the advancement of algebra. In pairs, compare your answers.

1. **When did the pursuit of finding solutions to equations begin?**
   a. the 3rd century B.C.E.
   b. 1650 B.C.E.
   c. 825 C.E.
   d. the 19th century
2. **What method did the ancient mathematicians, including the Babylonians, use to solve equations?**
   a. algebraic manipulation
   b. false position
   c. completing the square
   d. abstract reasoning
3. **What mathematical challenge did the Greeks face when dealing with cubic equations?**
   a. difficulty in solving linear equations
   b. trouble in geometric constructions for cubic products

c. lack of symbolic representation
d. inability to work with negative numbers

4. When did symbols start being used in algebraic problems?

a. the 3rd century
b. 825 C.E.
c. the 16th century
d. the 19th century

5. Who is credited with making a significant step towards the development of modern algebra in the 9th century?

a. Fibonacci
b. Diophantus
c. Muhammad ibn Musa al-Khwarizmi
d. Euclid

6. What did al-Khwarizmi's work lead to, influencing the study of algebra in Europe?

a. the spread of symbolic logic
b. the cossic art
c. quadratic equation solutions
d. the Rhind papyrus

7. Which Renaissance scholar published solutions to cubic and quartic equations in "The Great Art"?

a. René Descartes
b. Girolamo Cardano
c. Scipione del Ferro
d. Niccolò Tartaglia

8. What did Carl Friedrich Gauss prove in 1797, shaping the understanding of algebra?

a. the commutative property
b. the fundamental theorem of algebra
c. the existence of irrational numbers
d. the general formula for quintic equations

9. What important characteristic of algebra changed with the introduction of abstract algebraic systems?

a. commutative multiplication
b. geometric proofs
c. symbolic notation
d. negative numbers

10. Who made significant contributions to the understanding of noncommutative algebras in the late 19th to early 20th century?

a. René Descartes
b. Carl Friedrich Gauss
c. Amalie Noether
d. Paolo Ruffini

Activity 82. Read the article. Review your answers to the quiz in Activity 81.

Finding solutions to equations is a pursuit that dates back to the ancient Egyptians and Babylonians and can be traced through the early Greeks' mathematics. The Rhind papyrus, dating from around 1650 B.C.E., for instance, contains a problem reading:

A quantity; its fourth is added to it. It becomes fifteen. What is the quantity?

Readers are advised to solve problems like these by a method of "false position," where one guesses (posits) a solution, likely to be wrong, and adjusts the guess according to the result obtained. In this example, to make the division straightforward, one might guess that the quantity is four. Taking 4 and adding to it its fourth gives, however, only 4 + 1 = 5, one-third of the desired answer of 15. Multiplying the guess by a factor of three gives the solution to the problem, namely 4 × 3, which is 12.

Although the method of false position works only for linear equations of the form ax = b, it can nonetheless be an effective tool. In fact, several of the problems presented in the Rhind papyrus are quite complicated and are solved relatively swiftly via this technique.

Clay tablets dating back to 1700 B.C.E. indicate that Babylonian mathematicians were capable of solving certain quadratic equations by the method of completing the square. They did not, however, have a general method of solution and worked only with a set of specific examples fully worked out. Any other problem that arose was matched with a previously solved example, and its solution was found by adjusting the numbers appropriately.

Much of the knowledge built up by the old civilizations of Egypt and Babylonia was passed on to the Greeks. They took matters in a different direction and began examining all problems geometrically by interpreting numbers as lengths of line segments and the products of two numbers as areas of rectangular regions. Followers of Pythagoras from the period 540 to 250 B.C.E., for instance, gave geometric proofs of the distributive property and the difference of two squares formula, for example, in much the same geometric way we use today to explain the method of expanding brackets. The Greeks had considerable trouble solving cubic equations, however, since their practice of treating problems geometrically led to complicated three-dimensional constructions for coping with the product of three quantities.

At this point, no symbols were used in algebraic problems, and all questions and solutions were written out in words (and illustrated in diagrams). However, in the 3rd century, Diophantus of Alexandria introduced the idea of abbreviating the statement of an equation by replacing frequently used quantities and operations with symbols as a kind of shorthand. This new focus on symbols had the subtle effect of turning Greek thinking away from geometry. Unfortunately, the idea of actually using the symbols to solve equations was ignored until the 16th century.

The Babylonian and Greek schools of thought also influenced the development of mathematics in ancient India. The scholar Brahmagupta (ca. 598–665) gave solutions to quadratic equations and outlined general methods for solving systems of equations containing several variables. (He also had a clear understanding of negative numbers and was comfortable working with zero as a valid numerical quantity.) The scholar Bhaskara (ca. 1114–85) used letters to represent unknown quantities and, in working with quadratic equations, suggested that all positive numbers have two square roots and that negative numbers have no (meaningful) roots.

A significant step toward the development of modern algebra occurred in Baghdad, Iraq, in the year 825 when the Arab mathematician Muhammad ibn Musa al-Khwarizmi (ca. 780–850) published his famous piece “Calculation by Restoration and Reduction”. This work represents the first clear and complete exposition on the art of solving linear equations by a new practice of performing the same operation on both sides of an equation. For example, the expression $x – 3 = 7$ can be “restored” to $x = 10$ by adding three to both sides of the expression, and the equation $5x = 10$ can be “reduced” to $x = 2$ by dividing both sides of the equation by five. Al-Khwarizmi also showed how to solve quadratic equations via similar techniques. His descriptions, however, used no symbols, and like the ancient Greeks, al-Khwarizmi wrote everything out in words. Nonetheless, al-Khwarizmi’s treatise was enormously influential, and his new approach to solving equations paved the way for modern algebraic thinking. In fact, it is from the word “al-jabr” in the title of his book that our word “algebra” is derived.

Al-Khwarizmi’s work was translated into Latin by the Italian mathematician Fibonacci (ca. 1175–1250), and his efficient methods for solving equations quickly spread across Europe during the 13th century. The art of algebra became known in Europe as “the cossic art” (from the Italian word “cosa” for “thing”).

Renaissance scholars Scipione del Ferro (1465–1526) and Niccolò Tartaglia (ca. 1500–57) both knew how to solve cubic equations. In 1545 Girolamo Cardano (1501–76) published “The Great Art”, which included solutions to the cubic and quartic equations discovered by his assistant Ludovico Ferrari (1522–65).

By the end of the 17th century, mathematicians were comfortable performing the same sort of symbolic manipulations we practice today and were willing to accept negative numbers and irrational quantities as solutions to equations. The French mathematician François Viète (1540–1603) introduced an efficient system for denoting powers of variables and was the first to use letters as coefficients before variables, as in $ax^2 + bx + c$, for instance. (Viète also introduced the signs + and –, although he never used a sign for equality.) René Descartes (1596–1650) introduced the convention of denoting unknown quantities by the last letters of the alphabet, “x”, “y”, and “z”, and known quantities by the first, “a”, “b”, “c”. (This convention is now completely ingrained; when we see, for example, an equation of the form $ax + b = 0$, we assume, without question, that it is for “x” we must solve.)

The German mathematician Carl Friedrich Gauss (1777–1855) proved the fundamental theorem of algebra in 1797, which states that every polynomial equation of degree "n" has at least one and at most "n" (possibly complex) roots. His work, however, does not provide actual methods for finding these roots.

For the centuries that followed, mathematicians attempted to find a general arithmetic method for solving all quintic (fifth-degree) equations. Leonhard Euler (1707–83) suspected that the task might be impossible. Between the years 1803 and 1813, the Italian mathematician Paolo Ruffini (1765–1822) published a number of algebraic results that strongly suggested the same, and just a few years later the Norwegian mathematician Niels Henrik Abel (1802–29) proved that, indeed, there is no general formula that solves all quintic equations in a finite number of arithmetic operations. Of course, some degree-five equations can be solved algebraically. (Equation of the form $x^5 - a = 0$, for instance, have solutions $\sqrt[5]{x} = \sqrt{a}$.) In 1831 the French mathematician Évariste Galois (1811–32) completely classified those equations that can be so solved, developing work that gave rise to a whole new branch of mathematics today called group theory.

In the 19^{th} century mathematicians began using variables to represent quantities other than real numbers. For example, English mathematician George Boole (1815–64) invented an algebra of symbolic logic in which variables represented sets, and the Irish scholar Sir William Rowan Hamilton (1805–65) invented algebraic systems in which variables represented vectors or quaternions.

With these new systems, important characteristics of algebra changed. Hamilton, for instance, discovered that multiplication was no longer commutative in his systems: a product a × b might not necessarily give the same result as b × a. This motivated mathematicians to develop abstract axioms to explain the workings of different algebraic systems. Thus, the topic of abstract algebra was born. One outstanding contributor in this field was German mathematician Amalie Noether (1883–1935), who made important discoveries about the nature of noncommutative algebras.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 83. Identify the individuals based on the descriptions from the text in Activity 82.

Amalie Noether / Bhaskara / Brahmagupta / Carl Friedrich Gauss / Diophantus of Alexandria / Évariste Galois / Fibonacci / François Viète / George Boole / Girolamo Cardano / Leonhard Euler / Ludovico Ferrari / Muhammad ibn Musa al-Khwarizmi / Niccolò Tartaglia / Niels Henrik Abel / Paolo Ruffini / René Descartes / Scipione del Ferro / Sir William Rowan Hamilton

1. The Arab mathematician who published "Calculation by Restoration and Reduction," pioneering modern algebraic thinking.
2. The assistant to Girolamo Cardano, discovered solutions to cubic and quartic equations.
3. An English mathematician who invented an algebra of symbolic logic.
4. A French mathematician who classified equations solvable by algebraic methods, contributing to group theory.
5. A French mathematician who introduced an efficient system for denoting powers of variables.
6. A German mathematician who made important discoveries about the nature of noncommutative algebras.
7. A German mathematician who proved the fundamental theorem of algebra.
8. Introduced the convention of denoting unknown quantities by the last letters of the alphabet.
9. Introduced the idea of abbreviating the statement of an equation with symbols.
10. An Irish scholar who invented algebraic systems with variables representing vectors or quaternions.
11. An Italian mathematician who published results suggesting the impossibility of a general formula for quintic equations.
12. An Italian mathematician who translated al-Khwarizmi's work into Latin, spreading algebraic methods across Europe.
13. A Mathematician who suspected the impossibility of finding a general arithmetic method for solving all quintic equations.
14. A Norwegian mathematician who proved the impossibility of a general formula for quintic equations.
15. Published "The Great Art," including solutions to cubic and quartic equations.
16. A Renaissance scholar who knew how to solve cubic equations.
17. A scholar who gave solutions to quadratic equations and outlined general methods for solving systems of equations.
18. A scholar who used letters to represent unknown quantities and made contributions to quadratic equations.

Activity 84. Rearrange the events in chronological order according to the text of Activity 82. Provide dates where possible.

a. Amalie Noether makes significant discoveries about noncommutative algebras.
b. Babylonian mathematicians solve quadratic equations using the method of completing the square.
c. Carl Friedrich Gauss proves the fundamental theorem of algebra.
d. Diophantus introduces the idea of using symbols to abbreviate equations.
e. Equations are solved using the method of "false position" on the Rhind papyrus.
f. Fibonacci translates al-Khwarizmi's work into Latin, spreading algebraic methods in Europe.
g. François Viète introduces efficient systems for denoting powers of variables, and René Descartes establishes conventions for denoting known and unknown quantities.
h. Greeks interpret numbers geometrically, using lengths of line segments and areas of rectangular regions.
i. Mathematicians, including Paolo Ruffini and Niels Henrik Abel, explore general methods for solving quintic equations.
j. Mathematicians, such as George Boole and Sir William Rowan Hamilton, introduce algebraic systems representing sets, vectors, and quaternions.
k. Muhammad ibn Musa al-Khwarizmi publishes "Calculation by Restoration and Reduction," introducing algebraic methods in solving linear equations.
l. Scholars like Scipione del Ferro, Niccolò Tartaglia, and Girolamo Cardano contribute to solving cubic and quartic equations.

Activity 85. Determine whether the statements are true or false by quoting from the text in Activity 82.

1. Al-Khwarizmi did not heavily rely on symbols in his descriptions, similar to the ancient Greeks.
2. Babylonian mathematicians employed the technique of completing the square to find solutions for specific quadratic equations.
3. Babylonian mathematicians lacked a universal approach to solve various quadratic equations.

4. Fibonacci invented algebraic systems involving variables representing vectors or quaternions.
5. George Boole introduced the convention of using the first letters of the alphabet for denoting unknown quantities.
6. In 1797, Carl Friedrich Gauss proved the fundamental theorem of algebra.
7. In the 3rd century, Diophantus of Alexandria brought in the practice of using symbols to condense the expression of an equation.
8. Symbols were not commonly utilized by Greek mathematicians in their algebraic problem-solving.
9. The Italian mathematician Fibonacci translated the writings of al-Khwarizmi into Greek, disseminating algebraic techniques throughout Europe.
10. The work of Muhammad ibn Musa al-Khwarizmi in 825 played a crucial role in shaping modern algebraic thought.

Activity 86. In groups, discuss the points. Refer to the text in Activity 82.

1. Discuss the historical methods employed by ancient Egyptians and Babylonians, as seen in the Rhind papyrus, for finding solutions to equations. How effective was the method of "false position" in solving complex problems?
2. Explore the mathematical contributions of Babylonian mathematicians in solving quadratic equations using the method of completing the square. How did their approach differ from the methods used by other ancient civilizations?
3. Examine the geometric focus of Greek mathematicians, particularly followers of Pythagoras, in solving problems. How did their emphasis on geometry present challenges in dealing with cubic equations?
4. Analyze the transition from geometric problem-solving to symbolic notation introduced by Diophantus of Alexandria in the 3rd century. How did the use of symbols impact Greek thinking and the solving of equations?
5. Investigate the pivotal role played by Muhammad ibn Musa al-Khwarizmi in the development of modern algebra. How did his work in Baghdad in 825 lay the foundation for solving linear and quadratic equations?
6. Trace the spread of algebraic methods in Europe during the 13th century, facilitated by the translation of al-Khwarizmi's work by Fibonacci. How did this dissemination contribute to the recognition of algebra as "the cossic art"?
7. Examine the advancements made by Renaissance scholars like Scipione del Ferro, Niccolò Tartaglia, and Girolamo Cardano in solving cubic and quartic equations. How did their contributions shape the understanding of algebra in the 16th century?
8. Discuss the evolution of symbolic manipulations and the acceptance of negative numbers and irrational quantities in the 17th century. How did mathematicians like François Viète and René Descartes contribute to these developments?

9. Explore Carl Friedrich Gauss's proof of the fundamental theorem of algebra in 1797. How did this theorem impact the understanding of polynomial equations, and what were its implications?
10. Investigate the attempts to find a general arithmetic method for solving quintic equations in the 19th century. How did mathematicians like Paolo Ruffini, Niels Henrik Abel, and Évariste Galois contribute to the classification of solvable equations and the emergence of group theory?

Activity 87. Choose one point in Activity 86 and elaborate on it in writing. Refer to the text in Activity 82.

"There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world." (Nikolai Lobachevsky)