"There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world." (Nikolai Lobachevsky)

Module 3. Geometry

Unit 11. Analytic Geometry

Activity 88. Add the missing forms in the table and complete the sentences with the words.

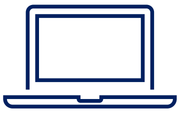
Noun	**Noun (Person)**	**Adjective**	**Adverb**	**Verb**
(singular) analysis (plural)		1) 2)		

1. A skilled mathematician acts as an __________, utilizing mathematical tools and methods to dissect complex problems and provide solutions.
2. An __________ approach in mathematics involves breaking down complex problems into simpler components for a more detailed examination.
3. Collaborative efforts among __________ are common in mathematical research, where multiple perspectives contribute to a more comprehensive understanding of a problem.
4. In __________ geometry, mathematical relationships between geometric shapes are explored using coordinate systems, providing a powerful tool to study the properties of curves, lines, and shapes through algebraic equations.
5. Mathematicians conduct rigorous __________ of mathematical structures to gain deeper insights into their properties and behaviour.
6. Mathematicians often __________ data sets, functions, and mathematical structures to extract meaningful information and draw relevant conclusions.
7. The __________ skills of mathematicians are crucial for interpreting data, formulating theorems, and developing precise solutions to mathematical challenges.
8. The ability to __________ mathematical concepts from different angles is a hallmark of a proficient mathematician, leading to a more comprehensive understanding of the subject.

Activity 89. In pairs, discuss the questions.

1. What was René Descartes?
2. What is the origin of the word “Cartesian”?
3. What is René Descartes’ greatest contribution to mathematics?
4. What is the difference between analytic geometry and coordinate geometry?

Figure 5. René Descartes (by Frans Hals)



Activity 90. Label the elements of the graph in Figure 6.

1) x, y
2) I, II, III, IV
3) (0, 0)
4) (2, 3)

a) axis
b) ordered pair of point coordinates
c) origin
d) quadrant

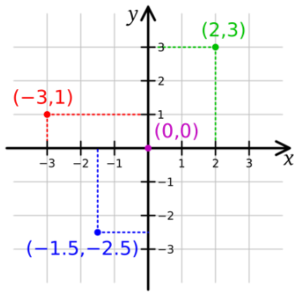


Figure 6. Coordinate Plane

Activity 91. Divide the text into paragraphs.

A straight line, usually horizontal, for which each point on the line represents a real number R is called a number line (real line). One assumes that the line extends indefinitely both to the left and to the right. A single point O on the line, called the origin, corresponds to the number zero in the real number system, and it is conventional to assume that a point “a” distance “a” units to the right of O represents the positive real number “a” and a point “b” units to the left of O the negative real number –b. The integers are thus represented as evenly spaced points, one unit apart, along the line. A number line is a one-dimensional Cartesian coordinate system (rectangular coordinate system, orthogonal coordinate system). The theory of cardinality shows that there are just as many points on the number line as there are points in a two-dimensional plane. The diagonal argument shows that the set of rational numbers (fractions) take up absolutely no space on the number line. A set of numbers used to locate a point on a number line, in a plane, or in space are called the coordinates of that point. For example, the coordinates of points on a number line could be given by their distances from a fixed point O, with points on one specified side of O being deemed a positive distance from O, and the points on the opposite side of O a negative distance from O. One way of assigning coordinates to points in the plane is to establish a fixed point O in the plane, and two lines of reference (called axes) that pass through O. Each axis is divided into a positive side and a negative side by O. Given a point P in the plane, one draws lines through P parallel to each of the axes. The distances along which these new lines intersect the axes specify the location of the point P. When the axes are drawn at right angles, the system is called a Cartesian coordinate system. The axes are usually called the x- and y-axes, and the pair of numbers (x,y) specifying the location of a point P (as "x" units along one axis, and "y" units along the second) are called the Cartesian coordinates of P. In three-dimensional space, the location of points can be specified via three mutually perpendicular (or oblique) axes passing through a common point O. The idea of assigning sets of numbers to points to specify locations is an old one. By the 3rd century B.C.E., Greek scholars Apollonius of Perga and Archimedes of Syracuse had used longitude, latitude, and altitude to define the position of a point on the Earth's surface. Roman and Greek surveyors labelled maps with grid lines, so as to specify locations via row and column numbers. One of the biggest breakthroughs in the development of mathematics occurred when geometry and algebra were united through the invention of the Cartesian coordinate system. Credited to 17th-century French mathematician and philosopher René Descartes (whose name Latinized reads Cartesius), Cartesian coordinates (rectangular coordinates, orthogonal coordinates) provide a means of representing each point in the plane via a pair of numbers. One begins by selecting a fixed point O in the plane, called the origin, and drawing through it two perpendicular number lines, called axes, one horizontal and one vertical, and both with the point O at the zero position on the line. It has become the convention to set the positive side of the horizontal number line to the right of O, and the positive side of the vertical number line above O, and to call the horizontal axis the x-axis, and the vertical one the y-axis. The Cartesian coordinates of a point P in the plane is a pair of numbers (x,y) which then describes the location of that point as follows: *The x-coordinate, or "abscissa," is the horizontal distance of the point from O along the horizontal axis. (A positive distance represents a point to the right of the vertical axis; a negative distance one to the left.) The y-coordinate, or "ordinate," is the vertical distance of the point from O along the vertical axis. (A positive distance represents a point located above the horizontal axis, and a negative distance one located below.)* Extending this idea to three-dimensions, points in space can be specified by a triple of numbers (x,y,z) representing the distances along three mutually perpendicular number lines. The coordinate axes are then called the x-, y-, and z-axes. They intersect at a point O, which is zero on all three number lines. The axes could be oriented to either form a left-handed or a right-handed system. The advent of a coordinate system allowed mathematicians, for the first time, to bring the power of algebra to the study of geometry. The French mathematician Nicole Oresme (1323–82) was the first to describe a way of graphing the relationship between an independent variable and a dependent one, and thus the first to make steps toward uniting geometry and algebra. The explicit construction of what we would call a coordinate system first appeared with the work of the

French lawyer and amateur mathematician Pierre de Fermat (1601–65). Starting with some horizontal reference line to represent an independent variable “x”, Fermat would graphically depict the relationship of a second variable “y” to it as a line segment, held at a fixed angle to the reference line, whose length would vary according to the variable “y” as it slides along the x-axis. Fermat did not think in terms, however, of identifying a second axis, nor did he require the line segment representing “y” to be perpendicular to the x-axis. In his famous 1637 text “Geometry”, René Descartes independently described similar methods for representing algebraic relationships graphically. Because the work of Fermat was not published until after his death, the discovery of coordinate geometry was attributed to Descartes. Because Fermat and Descartes interpreted the unknown variable “y” in an algebraic relationship as a physical length, both scholars only ever considered positive coordinates. The English mathematician John Wallis (1616–1703) was the first to introduce the possibility of negative coordinates. The idea of setting a fixed second axis, the y-axis, perpendicular to the x-axis was not popular until the mid 1700s. It was an idea that seemed to evolve gradually.

(from Elementary Geometry for College Students)

Activity 92. Based on Activity 91, for each paragraph, identify the topic sentence and provide a heading.

Activity 93. In groups, discuss the points. Refer to the text in Activity 91.

1. How did the concept of assigning coordinates to points evolve over time, from the number line to the Cartesian coordinate system? Discuss the contributions of Greek scholars, the role of maps, and the breakthroughs by mathematicians like Descartes.
2. Explore the impact of the Cartesian coordinate system on the development of mathematics. Discuss how the union of geometry and algebra changed the way mathematicians approached problem-solving and representation.
3. Compare the ancient methods of using longitude, latitude, and altitude to define a point's position with the Cartesian coordinate system. How did the understanding of coordinates in ancient times differ from the modern approach?
4. Examine the introduction of negative coordinates by John Wallis and the initial reluctance to consider them. Discuss why the concept of negative coordinates was ground-breaking and how it expanded the scope of coordinate geometry.
5. Explore the conventions established in the Cartesian coordinate system, such as positioning the positive side of the horizontal number line to the right and the positive side of the vertical number line above. Discuss the reasons behind these conventions and their importance in standardizing notation.

Activity 94. Single out the keywords (key phrases) from the text in Activity 91 and write an abstract of it.

Unit 12. Euclidean Geometry

Activity 95. Match the words with the definitions.

1. point
2. line
3. plane
4. angle
5. figure

a. a flat surface in which a straight line joining any two of its points lies entirely on that surface
b. a geometric element having no dimensions and whose position in space is located by means of its coordinates
c. any combination of points, lines, curves, or planes; a surface or space bounded on all sides by lines or planes
d. any straight one-dimensional geometrical element whose identity is determined by two points
e. the space between two straight lines that diverge from a common point or between two planes that extend from a common line

Activity 96. Classify the concepts under the headings. Represent the concepts of one class graphically.

area / broken / congruence / corner / curve / diagonal / edge / endpoint / face / height / horizontal / midpoint / parallel / perimeter / perpendicular / polygon / polyhedron / ray / segment / side / similarity / transversal / vertex / vertical / volume			
Line	**Position**	**Figure**	**Measure**

Activity 97. In pairs, discuss the questions.

1. Who is known as the “father of geometry”?
2. What is “The Elements”?
3. What is meant by Euclidean geometry?
4. Is all geometry Euclidean?

Activity 98. Read the article to examine the enduring impact of Euclid's "The Elements" on logic, mathematics, and education.

The branch of mathematics concerned with the properties of space and of figures, lines, curves, and points drawn in space is called geometry. Plane geometry examines objects drawn in a plane (lines, circles, polygons, and the like), solid geometry, or stereometry, deals with figures in three-dimensional space (polyhedra, lines, planes, and surfaces), and spherical geometry studies the properties of lines and shapes drawn on the surface of a sphere. The word "geometry" comes from the Greek words "ge" meaning "earth" and "metria" meaning "measure." As the origin of the word suggests, the study of geometry evolved from very practical concerns with regard to the accurate measurement of tracts of land, navigation, and architecture.

The geometry based on the definitions and axioms set out in Euclid's famous work "The Elements" is called Euclidean geometry. The salient feature of this geometry is that the fifth postulate, the parallel postulate, holds. It follows from this that through any point in the plane there is precisely one line through that point parallel to any given direction, that all angles in a triangle sum to precisely 180°, and that the ratio of the circumference of any circle to its diameter is always the same value π. Two-dimensional Euclidean geometry is called plane geometry, and the three-dimensional Euclidean geometry is called solid geometry. In 1899 German mathematician David Hilbert (1862–1943) proved that the theory of Euclidean geometry is free from contradiction.

Euclid of Alexandria (ca. 300–260 B.C.E.) began his famous 13-volume piece "The Elements" with 23 definitions ("a point is that which has no part" and "a line is that which has no breadth") followed by 10 axioms divided into two types: five common notions and five postulates.

His common notions were:

1. *Things that are equal to the same thing are equal to one another.*
2. *If equal things are added to equals, then the wholes are equal.*
3. *If equal things are subtracted from equals, then the remainders are equal.*
4. *Things that coincide with one another are equal to one another.*
5. *The whole is greater than the part.*

Euclid's postulates were:

1. *A straight line can be drawn to join any two points.*
2. *Any straight line segment can be extended to a straight line of any length.*
3. *Given any straight line segment, it is possible to draw a circle with centre one endpoint and with the straight line segment as the radius.*
4. *All right angles are equal to one another.*

5. *If two straight lines emanating from the endpoints of a given line segment have interior angles on one given side of the line segment summing to less than two right angles, then the two lines, if extended, meet to form a triangle on that side of the line segment.*

It is worth noting that Euclid deliberately avoided any direct mention of the notion of infinity. His wording of the second postulate, for instance, avoids the need to state that straight lines can be extended indefinitely, and his fifth postulate, also known as the parallel postulate, avoids direct mention of parallel lines, that is, lines that never meet when extended indefinitely.

From these basic assumptions Euclid deduced, by pure logical reasoning, 465 statements of truth (theorems) about geometric figures. The systematic approach he followed and the rigour of reasoning he introduced was hailed as a great intellectual achievement. His model of mathematical exploration became the standard for all mathematical research for the next 2,000 years.

Euclid's fifth postulate was always regarded with suspicion. It was never viewed as simple and as self-evident as his remaining four postulates, and Euclid himself did his utmost to avoid using it in his work. (Euclid did not invoke the fifth postulate until his 29th proposition.) Over the centuries scholars came to believe that the fifth postulate could be logically deduced from the remaining four postulates and therefore did not need to be listed as an axiom. Many people proposed proofs for it, including the 5th-century Greek philosopher Proclus, who is noted for his historical account of Greek geometry. Unfortunately, his proof was flawed, as were the proofs proposed by Arab scholars of the 8th and 9th centuries, and by Western scholars of the Renaissance.

In 1733 Italian teacher and scholar Girolamo Saccheri (1667–1733) believed that because Euclid's axioms model the real world, which he thought to be consistent, they cannot lead to a contradiction. If the first four postulates do indeed imply that the fifth postulate is also true, then assuming the four postulates together with the negation of the fifth postulate should lead to a logical inconsistency. Unfortunately, in following this tact, Saccheri never came across a contradiction.

In 1795 Scottish mathematician and physicist John Playfair (1748–1819) proposed an alternative formulation of the famous fifth postulate (today known as Playfair's axiom). It states:

In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.

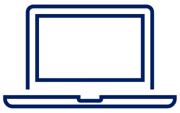
This version of the axiom is considerably easier to handle, and its negation is easier to envision. In an attempt to follow Saccheri's approach, Russian mathematician Nikolai Ivanovich Lobachevsky (1792–1856) and Hungarian mathematician János Bolyai (1802–1860), independently came to the same surprising conclusion: the first four of Euclid's postulates together with the negation of Playfair's version of the fifth postulate will not lead to a contradiction. This established, once and for all, that the fifth postulate is an independent axiom and cannot be deduced from the remaining four postulates. More important, by exploring the geometries that result in assuming that the fifth postulate does not hold, scholars were led to the discovery of non-Euclidean geometry.

In the late 1800s the German mathematician David Hilbert (1862–1943) noted that, despite its rigour, Euclid’s work contained many hidden assumptions. He also realized, despite Euclid’s attempts to describe them, that the notions of “point,” “line,” and “plane” cannot be properly defined and must remain as undefined terms in any theory of geometry. In his 1899 work “Foundations of Geometry” Hilbert refined and expanded Euclid’s postulates into a list of 28 basic assumptions that define all that is needed in a complete account of Euclid’s geometry. His axioms are today referred to as Hilbert’s axioms.

(from Elementary Geometry for College Students)

Table 28. Polygons

Polygon	**Edges Vertices**	**Polygon**	**Edges Vertices**	**Polygon**	**Edges Vertices**
		undecagon hendecagon	11		
		dodecagon	12	icosagon	20
triangle trigon	3	tridecagon triskaidecagon	13	triacontagon	30
quadrilateral tetragon	4	tetradecagon tetrakaidecagon	14	tetracontagon	40
pentagon	5	pentadecagon pentakaidecagon	15	pentacontagon	50
hexagon	6	hexadecagon hexakaidecagon	16	hexacontagon	60
heptagon	7	heptadecagon heptakaidecagon	17	heptacontagon	70
octagon	8	octadecagon octakaidecagon	18	octacontagon	80
nonagon enneagon	9	enneadecagon enneakaidecagon	19	enneacontagon	90
decagon	10			hectogon	100



Activity 99. Match the two parts of the sentences to classify angles.

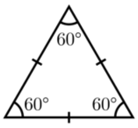
1. An angle of 0°
2. An angle between 0° and 90°
3. An angle of 90°
4. An angle between 90° and 180°
5. An angle of 180°
6. An angle between 180° and 360°
7. An angle of 360°

a. is called acute.
b. is called obtuse.
c. is called a reflex angle.
d. is called a round angle (or a perigon).
e. is called a straight angle.
f. is called a right angle.
g. is called a null angle.

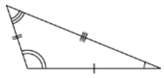
Activity 100. Label the triangles (trigons). Describe their geometric properties.

Lengths of Sides

1) a scalene triangle
2) an equilateral triangle
3) an isosceles triangle

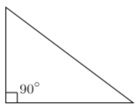
a) 

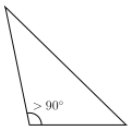
b) 

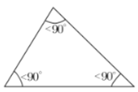
c) 

Internal Angles

4) a right triangle
   a right-angled triangle
5) an acute triangle
   an acute-angled triangle
6) an obtuse triangle
   an obtuse-angled triangle

d) 

e) 

f) 

Activity 101. Who was the first to prove Pythagoras' theorem? Watch the video "How Many Ways Are There to Prove the Pythagorean Theorem?" to choose the best answer to the questions. Then watch the video again and make a note of the famous proofs in the video. Offer more proofs.

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1. What do Euclid, young Einstein, and President Garfield have in common?
A. They all taught mathematics at universities
B. They all created proofs for the Pythagorean theorem
C. They all discovered new geometric shapes
D. They all studied ancient Babylonian mathematics

2. How did ancient Egyptian surveyors likely use the Pythagorean theorem in their work?
A. They used it to measure the height of pyramids
B. They used a knotted rope to create right angles for building
C. They calculated distances between cities
D. They designed irrigation systems for farms

3. What is the main purpose of mathematical proofs?
A. To make mathematics more difficult for students
B. To show that a theorem works for all cases, not just specific examples
C. To honor famous mathematicians from history
D. To create new mathematical rules and formulas

4. In the proof by rearrangement, what stays the same when the triangles are moved?
A. The shape of the overall figure and the area of the triangles
B. The angles of the triangles and the length of the sides
C. The total area of the figure and the area of the triangles
D. The position of the squares and the size of the hypotenuse

5. What can be inferred about the Pythagorean theorem?
A. It only works for triangles with specific measurements
B. It has inspired many different ways of proving the same mathematical truth
C. It was forgotten after Pythagoras died and rediscovered later
D. It is too complicated for most people to understand

Unit 13. Non-Euclidean Geometries

Activity 102. Label each geometric figure as flat (two-dimensional) or solid (three-dimensional). Add more figures of each type.

1. ball
2. circle
3. cone
4. cube
5. cylinder
6. disk
7. parallelepiped
8. parallelogram
9. polygon
10. polyhedron
11. prism
12. pyramid
13. quadrilateral
14. rectangle
15. rhombus
16. sphere
17. square
18. trapezium
19. trapezoid
20. triangle

Activity 103. Explain the concepts graphically in relation to the circle.

1. arc
2. centre
3. chord
4. circumference
5. diameter
6. origin
7. pi
8. quadrant
9. radius
10. secant
11. sector
12. segment
13. semicircle
14. tangent

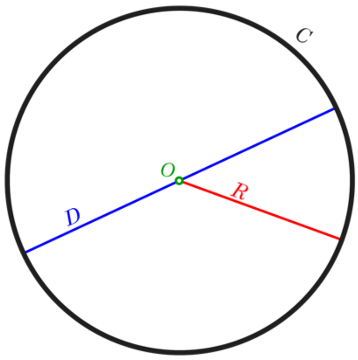

Figure 7. Circle

Activity 104. What are the three types of conic sections? Watch the video "The Mathematics of Sidewalk Illusions" to choose the best answer to the questions. Offer a geometric interpretation of illusory street art.

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1. What is anamorphosis?
A. A painting technique that uses only bright colors
B. A special type of perspective art that creates realistic 3D views on 2D surfaces
C. An ancient Roman method of drawing small figures

D. A mathematical formula used in modern computer graphics

2. What must a viewer do to see an anamorphic image correctly?
A. Look at it from multiple angles at the same time
B. Stand very close to the artwork with both eyes open
C. Position themselves in exactly the right spot
D. Use special glasses to see the hidden image

3. How did Leonardo da Vinci create the first known anamorphic drawing?
A. He painted directly onto a window at an angle
B. He used mathematical principles to manipulate perspective
C. He copied an ancient Greek technique
D. He projected light through colored glass

4. Why do parallel lines in perspective drawings need to converge to a vanishing point?
A. They must always be drawn at an angle to look realistic
B. All lines in art should meet at the center
C. They are only drawn parallel if they're parallel to the canvas plane
D. Renaissance artists preferred this artistic style

5. What is the main purpose of creating anamorphic sidewalk drawings?
A. To teach people about Renaissance art techniques
B. To create the illusion that a 3D image has been added seamlessly into an existing scene
C. To make exact copies of famous paintings on the ground
D. To show that sidewalks can be used as art galleries

Activity 105. Read the article to complete the table differentiating between the three geometries.

After numerous unsuccessful attempts throughout history to establish the parallel postulate as a consequence of the remaining four of Euclid's postulates, mathematicians began to contemplate theories of geometry in which the fifth postulate does not hold. Any such theory of geometry is called a non-Euclidean geometry.

In 1795 the Scottish mathematician and physicist John Playfair (1748–1819) presented an alternative, but equivalent, formulation of the parallel postulate:

Through any point in the plane, there is precisely one line through that point parallel to any prescribed direction.

Recasting the postulate this way makes it apparent that negation of the famous fifth postulate has two parts. Either:

1. *There are no lines through a given point parallel to a given direction.*
2. *There is more than one line through a given point parallel to a given direction.*

Independently discovered in 1823 by the Hungarian mathematician János Bolyai (1802–60) and in 1829 by the Russian mathematician Nikolai Ivanovich Lobachevsky (1792–1856), hyperbolic geometry (Lobachevskian geometry) is a non-Euclidean geometry in which the famous parallel postulate fails in the following manner:

Through a given point not on a given line, there is more than one line parallel to that given line.

The French mathematician Jules-Henri Poincaré (1854–1912) later provided a simple model for this geometry and the means to easily visualize geometric results in this theory. The "Poincaré disk" consists of all the points in the interior of the unit circle. A "point" in hyperbolic geometry is any point inside this circle, and a "line" is to be interpreted as a circular arc within the circle with endpoints perpendicular to the boundary of the circle. Any diameter of the boundary circle is also considered a line. Distances are not measured with a traditional ruler: points on the boundary circle are considered to be infinitely far from the centre of the circle.

Bolyai and Lobachevsky showed that all but the fifth of Euclid's postulates hold in the hyperbolic geometry and, moreover, that this model of geometry is consistent (that is, free of contradictions). This establishes that the parallel postulate cannot be logically deduced as a consequence of the remaining axioms proposed by Euclid.

In hyperbolic geometry, all angles in triangles sum to less than 180°, and the ratio of the circumference of any circle to its diameter is less than π. (Moreover, the value of this ratio is not the same for all circles.) Also, it is possible for two perpendicular lines to be parallel to the same line.

Physicists, following the work of Albert Einstein, suggest that the geometry of our universe is hyperbolic: that it appears to us as Euclidean is a result of the fact that we occupy such a small portion of it. (This is analogous to the fact that it is difficult to recognize the Earth as round when living on it.)

Discovered in 1856 by the German mathematician Georg Friedrich Bernhard Riemann (1826–66), and later slightly modified by Felix Klein (1849–1925), elliptic geometry (Riemannian geometry) is a non-Euclidean geometry in which the famous parallel postulate fails in the following manner:

Through a given point not on a given line, there are no lines parallel to that given line.

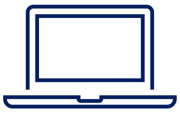
Riemann used the surface of a sphere as a model of this geometry by interpreting the word "line" to mean a great circle on the sphere. Given that in a theory of geometry two lines are meant to intersect at just one point (yet any two great circles intersect at two antipodal points), it is appropriate then to interpret the word "point" in elliptic geometry as an antipodal pair of points on the surface. In this setting, it is now also true that any two distinct points determine a unique line.

Riemann and Klein proved that all but the fifth of Euclid's postulates hold in this model and, moreover, that this model is consistent (that is, free of contradictions). This establishes that the parallel postulate cannot be logically deduced as a consequence of the remaining axioms proposed by Euclid.

In elliptic geometry all angles in triangles sum to more than 180°, and the ratio of the circumference of any circle to its diameter is greater than π (and this value varies from circle to circle).

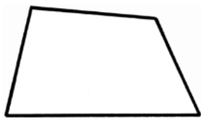
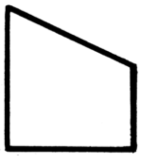
(from Elementary Geometry for College Students)

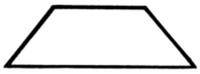
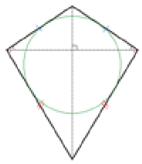
№	Geometry	Discoverer	Postulates	Properties
1	hyperbolic			
2	Euclidean			
3	elliptic			



Activity 106. Label the quadrilaterals (tetragons). Describe their geometric properties.

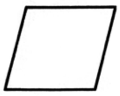
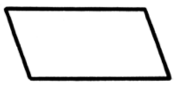
1) a kite, a deltoid
2) an irregular quadrilateral (BrE)
   a trapezium (AmE)
3) a trapezium (BrE)
   a trapezoid (AmE)
4) an isosceles trapezium (BrE)
   an isosceles trapezoid (AmE)

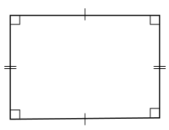
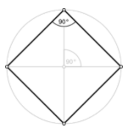
a)  b) 

c)  d) 

Parallelograms

5) a rectangle
6) a rhomboid
7) a rhombus, a rhomb, a diamond
8) a square

e)  f) 

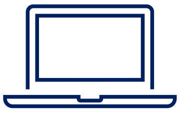
g)  h) 

Activity 107. Reorder the sentences to make a text on topology.

a. All that is required of a topological transformation is that points that begin close together remain relatively close together.
b. Any transformation that requires puncturing or tearing a surface, or joining together two disjoint portions of a figure, is not considered a valid topological action. (Tearing a surface, for instance, separates points that were close together.)
c. As it is impossible to deform a sphere into a torus without puncturing the surface, these two shapes are not topologically equivalent.
d. As other examples, the capital letters C, M, and Z are topologically equivalent, as are the letters D, O, P, and R, but no letter from the first group is topologically equivalent to a letter in the second.
e. For example, the statement, “Removing a point from a circle produces a line segment” is valid for the entire class of objects topologically equivalent to a circle.
f. In 1895 French mathematician Henri Poincaré (1854–1912) examined these works and published five papers, laying a theoretical framework for a formal study of topology.
g. In this viewpoint, a circle and a square (of any size) are topologically equivalent, since either can be continuously deformed into the other, and a number of geometrical properties apply equally well to either object.
h. No letter in the alphabet is topologically equivalent to the letter B (other than capital B itself).
i. The mathematician Carl Friedrich Gauss (1777–1855) examined the distortion of knots and invariant properties that arise in the study of projective geometry.
j. The study of those properties of geometric figures and surfaces that remain unchanged when the shapes of those objects are distorted by a continuous deformation (such as stretching, shrinking, or twisting) is called topology.
k. The study of topology began with Leonhard Euler (1707–83) and his analysis of the famous seven bridges of Königsberg problem.
l. Unlike a classical geometer, a topologist is not interested in questions of distances and angles but is only concerned with the relative positions of points.

Activity 108. Choose one geometric figure, flat or solid, to present. Demonstrate a method of its construction as well as a description of its geometric properties.

Unit 14. Development of Geometry



Activity 109. Complete the sentences with the opposites of the words in brackets by adding negative prefixes.

1. ___________ (Euclidean) geometry explores spaces with different postulates than classical geometry.
2. ___________ (numeracy) refers to a lack of basic numerical skills.
3. ___________ (rational) numbers cannot be expressed as fractions of integers.
4. Be cautious not to ___________ (calculate) the values in your test.
5. Despite the tutorial, some students remain ___________ (numerate) in basic calculations.
6. One is an ___________ (divisible) number in the realm of integers.
7. Solve the ___________ (equality) to determine the valid range of values.
8. The ___________ (calculation) in the initial step led to an incorrect result.
9. The ___________ (proper) fraction needed to be converted into a mixed number.
10. The complex ___________ (equation) involves variables on both sides of the equation.
11. The event is statistically ___________ (probable) given the current conditions.
12. The formula provided an ___________ (correct) solution to the mathematical problem.
13. The function exhibits ___________ (bounded) growth as x approaches infinity.
14. The lengths of the two lines are ___________ (equal) in this geometric figure.
15. The mathematical relationship remains ___________ (variable) under certain conditions.
16. The measurement was ___________ (exact) due to the limitations of the measuring tool.
17. The possibilities are ___________ (limited) when dealing with infinite sets.
18. The shape displays an ___________ (symmetrical) arrangement of elements.
19. The value of the variable remains ___________ (known) in this equation.

Activity 110. Do the quiz on the advancement of geometry. In pairs, compare your answers.

1. **When did Egyptian and Babylonian scholars develop principles of measurement and spatial reasoning?**
   a. 1900 B.C.E.
   b. 1650 B.C.E.
   c. 300–260 B.C.E.
   d. the 17th century
2. **What method did Egyptian scholars use to construct right angles?**
   a. algebraic equations
   b. the Pythagorean theorem
   c. knotted ropes
   d. trigonometric functions
3. **Who compiled a large volume of geometric knowledge in "The Elements" and introduced logical reasoning in geometry?**
   a. Isaac Newton
   b. Euclid
   c. René Descartes
   d. Carl Friedrich Gauss
4. **What breakthrough in geometry occurred in the 17th century with the work of René Descartes?**
   a. introduction of calculus
   b. discovery of non-Euclidean geometry
   c. use of Cartesian coordinates
   d. development of hyperbolic geometry
5. **Who pursued the task of developing a full algebraic model of geometry after Descartes?**
   a. Carl Friedrich Gauss
   b. Euclid
   c. John Playfair
   d. Pierre de Fermat
6. **Which mathematician inspired work on the theory of differential calculus through his contributions to geometry?**
   a. Albert Einstein
   b. Carl Friedrich Gauss
   c. Euclid

d. Pierre de Fermat

7. Who first permitted negative values for distances in coordinate geometry?

a. Sir Isaac Newton
b. Gottfried Wilhelm Leibniz
c. René Descartes
d. John Playfair

8. What did the Scottish mathematician John Playfair propose regarding Euclid's fifth postulate?

a. eliminating the fifth postulate
b. proving the fifth postulate as a consequence of the remaining four
c. introducing an equivalent postulate
d. revising the fifth postulate

9. What type of geometry did Nikolai Ivanovich Lobachevsky develop, challenging Euclid's fifth postulate?

a. Euclidean geometry
b. hyperbolic geometry
c. spherical geometry
d. elliptic geometry

10. What ground-breaking idea did Bernhard Riemann propose in his 1854 lecture?

a. the existence of parallel lines
b. the study of dimensions
c. non-Euclidean geometry
d. the foundations of calculus

Activity 111. Read the article. Review your answers to the quiz in Activity 110.

The study of geometry is an ancient one. Records show that Egyptian and Babylonian scholars of around 1900 B.C.E. had developed sound principles of measurement and spatial reasoning in their architecture and in their surveying of land. Both cultures were aware of Pythagoras's theorem and had developed tables of Pythagorean triples. (The Egyptians used knotted ropes to construct "3-4-5 triangles" to create right angles.) Ancient Indian texts on altar construction and temple building demonstrate sophisticated geometry knowledge, and the famous volume "The Nine Chapters on the Mathematical Art" from ancient China also includes work on the Pythagorean theorem.

In ancient Greece, mathematical scholars came to realize that many properties of shapes and figures could be deduced logically from other properties. In his epic work "The Elements" the Greek geometer Euclid (ca. 300–260 B.C.E.) collated a large volume of knowledge on the subject and showed that each and every result could be logically deduced from a very small set of basic assumptions (self-evident truths) about how geometry should work. Euclid's work was rigorous and systematic, and the notion of a logical proof was born. Euclid's postulates and the process of logical reasoning became the model of all further geometric investigation for the two millennia that followed. His method of compiling and organizing all mathematical knowledge known at his time was a significant intellectual achievement. Euclid's rigorous approach was, and still is, modelled in other branches of mathematics. Scholars in set theory, the foundations of mathematics, and calculus, for instance, all seek to follow the same process of formal reasoning as the correct approach to achieve proper understanding of these topics.

The next greatest breakthrough in the advancement of geometry occurred in the 17th century with the discovery of Cartesian coordinates as a means to represent points as pairs of real numbers and lines and curves as algebraic equations. This approach, described by the French mathematician and philosopher René Descartes (1596–1650) in his famous 1637 work "Geometry", united the then-disparate fields of algebra and geometry. Unfortunately, Descartes's interests lay only in advancing methods of geometric construction, not in developing a full algebraic model of geometry. This latter task was pursued by the French mathematician Pierre de Fermat (1601–65), who had also outlined the principles of coordinate geometry in an unpublished manuscript that he had circulated among mathematicians before the release of "Geometry". Fermat later published the work in 1679 under the title "On the Plane and Solid Locus". The application of algebra to the discipline provided scholars a powerful new tool for solving geometric problems, and also provided them with a large number of different types of curves for study.

Fermat's work in geometry inspired work on the theory of differential calculus and, later, led to the study of "differential geometry" (the application of calculus to the study of shapes and surfaces). This was developed by the German mathematician and physicist Carl Friedrich Gauss (1777–1855).

Neither Descartes nor Fermat permitted negative values for distances. Consequently, neither scholar worked with a full set of coordinate axes as we use them today. The notions of negative distance and negative area were first put forward by Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716), the coinventors of calculus.

The 19th century saw other major advances in geometry. It had long been noted that Euclid's fifth postulate, the so-called parallel postulate, is not necessary for a great deal of geometry. Many Arab scholars of the first millennium attempted, without success, to show that the fifth postulate could be logically deduced from the remaining four (thereby rendering it unnecessary), as did European scholars of the Renaissance. In 1795 the Scottish mathematician John Playfair showed that the fifth postulate is equivalent to the statement that, through any point, one can draw one, and only one, line through that point parallel to a given line. (This is today called Playfair's axiom.) Although not eliminating the need for the fifth postulate, Playfair showed that it could be understood in a more tractable form.

In 1829 the Russian mathematician Nikolai Ivanovich Lobachevsky (1792–1856) took a bold step and considered a geometric world in which the fifth postulate is false. He assumed that through a given point more than one line could be drawn parallel to a given line. In doing this, Lobachevsky discovered a new, consistent mathematical system free from contradiction, one as logically valid as the geometry of Euclid. (This geometry is today called hyperbolic geometry.) The philosophical impact of Lobachevsky's work was enormous: he had shown that mathematics need not be based on a single set of physical truths, and that other equally valid mathematical systems do exist based on alternative, carefully chosen axioms. Lobachevsky had also shown that Euclid's fifth postulate cannot be established as a consequence of the remaining four axioms: he had presented a valid example of a system in which the first four of Euclid's postulates hold, but the fifth does not.

Surprisingly some of Lobachevsky's ideas were anticipated well before the 19^{th} century. The great Persian mathematician and poet Omar Khayyam (ca. 1048–1122) established a number of results that we recognize today as non-Euclidean. These results were later translated into Latin, and extended upon, by the Italian priest Girolamo Saccheri (1667–1733). Unfortunately, neither scholar discovered the validity of non-Euclidean geometry, as each was focused instead on trying to establish Euclid's fifth postulate as a consequence of the remaining four.

The German mathematician Bernhard Riemann (1826–66) discovered an alternative form of non-Euclidean geometry in which Euclid's fifth postulate fails in a different way. In a system of spherical geometry, it is never possible to draw a line through a given point parallel to a given line.

Riemann's contributions to the advancement of geometry were significant. In his famous 1854 lecture "On the Hypotheses that Lie at the Foundation of Geometry", Riemann put forward the view that geometry can be the study of any kind of space of any number of dimensions, and later developed the mathematics needed to properly describe the shape of space. Albert Einstein (1879–1955) later used this work to develop his theory of relativity.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 112. Identify the individuals based on the descriptions from the text in Activity 111.

Albert Einstein / Bernhard Riemann / Carl Friedrich Gauss / Euclid / Girolamo Saccheri / Gottfried Wilhelm Leibniz / John Playfair / Nikolai Ivanovich Lobachevsky / Omar Khayyam / Pierre de Fermat / René Descartes / Sir Isaac Newton

1. The coinventor of calculus, who first permitted negative values for distances and introduced the notions of negative distance and negative area.
2. A French mathematician and philosopher who introduced Cartesian coordinates in his work "Geometry," uniting algebra and geometry.
3. A French mathematician who outlined the principles of coordinate geometry and applied algebra to solve geometric problems.
4. A German mathematician and physicist who developed "differential geometry," applying calculus to the study of shapes and surfaces.
5. A German mathematician who discovered an alternative form of non-Euclidean geometry (spherical geometry) and proposed that geometry can be the study of any space of any number of dimensions.
6. A Greek geometer who compiled "The Elements," a comprehensive work on geometry, and established logical proofs as a model for geometric investigations.
7. An Italian priest who translated and extended upon Omar Khayyam's results, focused on trying to establish Euclid's fifth postulate as a consequence of the remaining four.
8. A Persian mathematician and poet who established results recognized today as non-Euclidean, although he did not discover the validity of non-Euclidean geometry.
9. A physicist who used Riemann's work to develop his theory of relativity.
10. A Russian mathematician who developed hyperbolic geometry by considering a geometric world in which Euclid's fifth postulate is false.
11. A Scottish mathematician who, in 1795, showed the equivalence of the parallel postulate to the statement that through any point, one can draw one, and only one, line through that point parallel to a given line (now called Playfair's axiom).

Activity 113. Rearrange the events in chronological order according to the text of Activity 111. Provide dates where possible.

a. Albert Einstein's use of Riemann's work in the development of the theory of relativity.
b. Bernhard Riemann's discovery of an alternative form of non-Euclidean geometry.

c. Carl Friedrich Gauss's contributions to the theory of differential calculus and differential geometry.
d. Development of "The Nine Chapters on the Mathematical Art" in ancient China.
e. Development of hyperbolic geometry by Nikolai Ivanovich Lobachevsky.
f. Discovery of Cartesian coordinates and representation of points as pairs of real numbers.
g. Introduction of negative values for distances by Sir Isaac Newton and Gottfried Wilhelm Leibniz.
h. Invention of algebraic methods for geometric construction by René Descartes.
i. John Playfair's demonstration of the equivalence of Euclid's fifth postulate to Playfair's axiom.
j. Omar Khayyam's establishment of results recognized today as non-Euclidean.
k. Publication of Euclid's "The Elements" and the introduction of logical reasoning in geometry.
l. Recognition of Pythagoras's theorem and the use of knotted ropes to create "3-4-5 triangles" by Egyptian scholars.
m. Pierre de Fermat's publication of "On the Plane and Solid Locus" and principles of coordinate geometry.
n. John Playfair's work on the parallel postulate.

Activity 114. Determine whether the statements are true or false by quoting from the text in Activity 111.

1. A significant advancement in geometry during the 17^{th} century was the introduction of Cartesian coordinates by René Descartes (1596–1650), which fused algebra and geometry and was further developed by Pierre de Fermat (1601–65).
2. Bernhard Riemann (1826–66) advanced non-Euclidean geometry by incorporating hyperbolic geometry, proposing that geometry could explore spaces of various dimensions, a concept later utilized by Albert Einstein in his theory of relativity.
3. Euclid, the Greek geometer from approximately 300 to 260 B.C.E., created "The Elements," emphasizing logical deductions from basic principles as the cornerstone of geometry for centuries.
4. Geometry only gained prominence in the 19^{th} century, marked by breakthroughs like Cartesian coordinates and hyperbolic geometry.
5. Girolamo Saccheri (1667–1733) recognized non-Euclidean geometry's validity while attempting to prove Euclid's fifth postulate.
6. John Playfair's axiom successfully removed the necessity of Euclid's fifth postulate, streamlining the principles of geometry.
7. Newton and Leibniz advocated for negative values in distances, reshaping the use of coordinate axes.

8. Nikolai Ivanovich Lobachevsky (1792–1856) introduced hyperbolic geometry by questioning Euclid's fifth postulate, establishing an internally consistent mathematical system distinct from Euclidean geometry.
9. René Descartes (1596–1650) prioritized creating a comprehensive algebraic model for geometry over enhancing methods of geometric construction.
10. The exploration of geometry has ancient origins, with Egyptian and Babylonian scholars around 1900 B.C.E. establishing fundamental principles of measurement and spatial reasoning.

Activity 115. In groups, discuss the points. Refer to the text in Activity 111.

1. Explore the insights into ancient civilizations, such as Egypt, Babylon, India, and China, and their sophisticated understanding of geometry principles, including Pythagoras's theorem.
2. Discuss the significant impact of Euclid's work in "The Elements," examining how his logical approach and postulates shaped geometric investigation for over two millennia.
3. Delve into the 17^{th}-century breakthrough of Cartesian coordinates, as described by Descartes, and its role in uniting algebra and geometry. Consider the subsequent developments by Fermat.
4. Examine the shift from geometric construction to an algebraic model of geometry initiated by Fermat and its application in solving geometric problems and studying various curves.
5. Explore how Fermat's work influenced the development of differential calculus and led to the study of differential geometry by Carl Friedrich Gauss in the 18^{th} century.
6. Discuss the introduction of negative values for distances by Newton and Leibniz, and how this departure from Descartes and Fermat influenced coordinate axes and geometric understanding.
7. Investigate the historical attempts by Arab and European scholars to logically deduce Euclid's fifth postulate, leading to Playfair's axiom and the recognition of alternative interpretations.
8. Explore Lobachevsky's bold departure from Euclidean geometry, introducing hyperbolic geometry and the philosophical impact of showing that mathematics can be based on different axioms.
9. Discuss how Omar Khayyam and Girolamo Saccheri anticipated non-Euclidean ideas, even though they did not fully recognize the validity of such geometry.
10. Investigate Riemann's contributions to spherical geometry and the subsequent use of his work by Albert Einstein in developing the theory of relativity, illustrating the interdisciplinary nature of geometry.

Activity 116. Choose one point in Activity 115 and elaborate on it in writing. Refer to the text in Activity 111.

Unit 15. Trigonometry

Activity 117. Complete the sentences with the derivatives of the words in brackets.

1. __________ (algebra) expressions and equations involve variables and constants, allowing for the generalization of mathematical relationships.
2. __________ (analyze) geometry combines algebraic and geometric techniques to study geometric shapes using coordinate systems.
3. __________ (arithmetic) calculations are essential in everyday tasks, such as budgeting and financial planning.
4. __________ (geometry) series have a common ratio between consecutive terms.
5. __________ (logic) reasoning is a fundamental aspect of mathematical thinking, guiding the deduction of conclusions based on established principles.
6. __________ (statistics) analyzing a set of data involves calculating measures such as mean, median, and standard deviation to understand the central tendency and variability within the dataset.
7. __________ (topology) spaces provide a framework for studying properties preserved under continuous transformations.
8. __________ (trigonometry) functions, such as sine and cosine, describe relationships between angles and sides in triangles.
9. A __________ (statistics) applies mathematical and statistical techniques to analyze and interpret data.
10. A coffee cup and a donut are considered __________ (topology) equivalent.
11. A data __________ (analyze) employs mathematical and statistical techniques to interpret complex datasets and extract meaningful insights.
12. Elementary __________ (arithmetic) involves basic operations like addition, subtraction, multiplication, and division.
13. Mathematical __________ (analyze) involves the study of limits, continuity, and derivatives, essential for understanding functions and their behaviour.
14. Scientists use __________ (statistics) methods to analyze experimental data and draw meaningful conclusions.
15. The sine of an angle in a right-angled triangle is calculated __________ (trigonometry) as the ratio of the opposite side to the hypotenuse.
16. When solving a quadratic equation, one can find the roots __________ (algebra) by applying the quadratic formula.

Activity 118. Read the passage. In pairs, discuss the questions in the box.

The word "trigonometry" comes from the Greek words "trigonon" ("triangle") and "metron" ("to measure"). Contrary to its name, the theory of trigonometry (informally abbreviated to "trig") is best motivated as a theory about circles, not triangles. (This, in fact, matches the historical development of the subject.) Trigonometry is the branch of mathematics concerned with specific functions of angles and their application to calculations. There are six functions of an angle commonly used in trigonometry. Their names and abbreviations are sine (sin), cosine (cos), tangent (tan), cotangent (cot), secant (sec), and cosecant (csc). These six trigonometric functions in relation to a right triangle are displayed in the figure. For example, the triangle contains an angle A, and the ratio of the side opposite to A and the side opposite to the right angle (the hypotenuse) is called the sine of A, or sin A; the other trigonometry functions are defined similarly. These functions are properties of the angle A independent of the size of the triangle, and calculated values were tabulated for many angles before computers made trigonometry tables obsolete. Trigonometric functions are used in obtaining unknown angles and distances from known or measured angles in geometric figures. Trigonometry developed from a need to compute angles and distances in such fields as astronomy, mapmaking, surveying, and artillery range finding. Problems involving angles and distances in one plane are covered in plane trigonometry. Applications to similar problems in more than one plane of three-dimensional space are considered in spherical trigonometry.

(from Encyclopaedia Britannica)

1. What is trig?
2. Does the etymology of the word "trigonometry" reveal its meaning?
3. What are the six common trigonometric functions?
4. What is the use of the Greek letter θ (theta) in trigonometry?
5. What are the application areas of trigonometry?

Activity 119. Match the trigonometric functions with the ratios.

1. sine (sin)
2. cosine (cos)
3. tangent (tan / tg)
4. cosecant (csc / cosec)
5. secant (sec)
6. cotangent (cot / ctg)

a. $\frac{adjacent}{hypotenuse}$

b. $\frac{adjacent}{opposite}$

c. $\frac{hypotenuse}{adjacent}$

d. $\frac{hypotenuse}{opposite}$

e. $\frac{opposite}{adjacent}$

f. $\frac{opposite}{hypotenuse}$

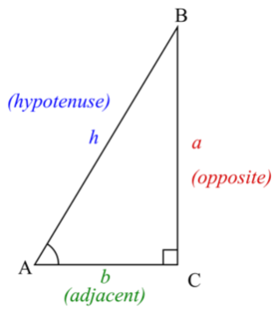


Figure 8. Right Triangle

Activity 120. What is the connection between navigation and trigonometry? Watch the video "How Does Math Guide Our Ships at Sea?" to choose the best answer to the questions Describe the function that trigonometry originally fulfilled.

https://disk.yandex.ru/i/YGDQZvzsvLow_Q

1. What was the main problem with dead reckoning as a navigation method?
A. It required too many sailors to operate the ship
B. Small errors in direction could cause ships to miss their destination
C. It only worked during daylight hours
D. It was too expensive for most ships to use

2. Why was it necessary to know the exact time back in England while at sea?
A. To calculate when the ship would arrive at its destination
B. To determine the ship's longitude by comparing sun angles
C. To know when to change the ship's direction
D. To record the ship's daily progress in the logbook

3. What was the main disadvantage of John Harrison's clock?
A. It wasn't accurate enough for navigation
B. It couldn't work in harsh sea conditions
C. It was very costly because it was handmade
D. It was too heavy to carry on most ships

4. How did Henry Briggs contribute to making logarithms more practical?
A. He invented logarithms in his castle in Scotland
B. He suggested changing the base to make calculations simpler
C. He created the first clock that could work at sea
D. He developed the sextant for measuring angles

5. What does the history of navigation demonstrate about creativity?
A. It shows that mathematics is the most important skill for innovation
B. It proves that working alone leads to the best inventions
C. It illustrates that collaboration between different fields is valuable
D. It suggests that expensive tools are necessary for progress

Activity 121. Do the quiz on trigonometry. In pairs, compare your answers.

1. **When did the use of triangles for determining distances, leading to the birth of trigonometry, begin?**
   a. the 2^{nd} century B.C.E.
   b. the 5^{th} century
   c. the 12^{th} century
   d. the 17^{th} century
2. **Which ancient Greek philosopher is credited with using similar triangles to determine the height of the Cheops pyramid?**
   a. Eratosthenes of Cyrene
   b. Hipparchus of Rhodes
   c. Thales of Miletus
   d. Claudius Ptolemy
3. **What did Greek astronomers choose to work with instead of angles when developing models for the motion of stars and planets?**
   a. sine values
   b. chords of circles
   c. tangent functions
   d. secant values
4. **In the 5^{th} century C.E., Indian scholars simplified the theory of chords by working with:**
   a. chord values
   b. sine ratios
   c. half-chords for half-angles
   d. tangent functions
5. **Which mathematician is believed to have systemized theorems and proofs of Indian trigonometry and possibly invented the tangent function?**
   a. Fibonacci
   b. Abu al-Wafa
   c. Georg Joachim Rheticus
   d. Leonhard Euler
6. **Who coined the terms "tangent" and "secant" and developed fundamental trigonometric formulae in the 17^{th} century?**
   a. Regiomontanus
   b. François Viète
   c. Thomas Fincke

d. Bartholomaeus Pitiscus

7. **When did the perspective of thinking about trigonometric quantities as ratios rather than actual line lengths emerge?**
    a. the 12^{th} century
    b. the 17^{th} century
    c. the 18^{th} century
    d. the 19^{th} century
8. **Which mathematician wrote "Introductio in analysin infinitorum" in 1748, outlining principles of trigonometry as we regard them today?**
    a. Regiomontanus
    b. Claudius Ptolemy
    c. François Viète
    d. Leonhard Euler

Activity 122. Read the article. Review your answers to the quiz in Activity 121.

From very early times, surveyors, architects, navigators, and astronomers have made use of triangles to determine distances that could not be measured directly. This gave birth to the subject we today know as trigonometry. There are problems in the ancient Egyptian text, the Rhind papyrus of around 1650 B.C.E., that call for the determination of the slope angles of pyramid faces using the equivalent of the cotangent function we use today, and a Babylonian clay tablet from around 1700 B.C.E. contains a table of secant values for the 15 angles between 30° and 45°. The Greek philosopher Thales of Miletus (ca. 600 B.C.E.) is said to have made use of similar triangles to determine the height of the Cheops pyramid by comparing the length of its shadow with the length of the shadow of a rod inserted in the ground. Eratosthenes of Cyrene (ca. 250 B.C.E.) computed the circumference of the Earth using lengths of shadows and a simple geometric argument on angles.

Greek astronomers of ancient times believed that the stars and planets of the night sky moved along circular arcs of a giant celestial sphere, and they worked to develop models that would accurately predict the motion of these objects on the sphere. Rather than phrase matters in terms of angles, which proved to be difficult, Greek astronomers chose to work with measures of straight lengths closely related to angles, namely, the lengths of chords of circles.

Hipparchus of Rhodes (ca. 200 B.C.E.) constructed a table of such chord lengths for a circle of circumference 21,600 = 360 × 60 units (which corresponds to 1 unit of circumference for each minute of arc).

In the 2^{nd} century C.E., the mathematician Claudius Ptolemy wrote the first extensive treatise on the theory of chords and their use in obtaining information about “spherical triangles,” that is, triangles made by great circular arcs on the surface of a sphere. In addition to working out theorems, Ptolemy explained how to construct tables of chord values, and presented his own list of chord values for all angles between zero and 180° in half-degree increments.

The next important step in the development of trigonometry occurred in India. Scholars of the 5th century C.E. had by this time discovered that working with half-chords for half-angles greatly simplified the theory of chords and its applications to astronomy. This approach is almost the same construct as the sine function of today. Whereas we think of sine as a ratio of lengths (the length of the half-chord to the radius), Indian scholars interpreted sine as the actual length of the half-chord. Of course, this value of an angle differed for circles of different sizes. The scholars Aryabhata, Bhaskara II, and others developed astonishingly sophisticated techniques for computing half-chord values.

The Arab scholars of the 10th century took a great interest in the work from India. The mathematician Abu al-Wafa (ca. 950) of Baghdad systemized theorems and proofs of Indian trigonometry and prepared his own comprehensive table of half-chord values. He is also believed to have invented the tangent function, which he called the "shadow," and possibly the secant and the cosecant functions. (Still, all were thought of as specific lengths, not as ratios of lengths.)

In the 12th century, European scholars began translating the Arabic works and soon became acquainted with the extensive theory of trigonometry. The famous scholar Fibonacci (ca 1170–1250) also travelled extensively in the Arab countries and wrote of their trigonometry in his famous work "Practica geometriae". Around 1464, the German astronomer and mathematician Regiomontanus (1436–76) compiled "De triangulis omnimodis", a compendium of trigonometry of that time. This work was enormously influential, and over the following decades other texts on the topic appeared. The German mathematician Georg Joachim Rheticus (1514–74) published, in 1551, "Canon doctrinae triangulorum", which defined, for the first time, all six basic trigonometric functions, and explained how to relate them to right triangles without making any reference to circles. The Danish physician Thomas Fincke (1561–1656) coined the terms "tangent" and "secant" and developed further fundamental trigonometric formulae. The word "trigonometry" itself was invented by the German mathematician Bartholomaeus Pitiscus (1561–1613). By this time, trigonometry was certainly regarded as a worthwhile topic of mathematical pursuit, independent of applications to astronomy.

The subject also proved to be useful in the study of algebra. The French mathematician François Viète (1540–1603) showed, for example, that one could solve certain cubic equations by making trigonometric substitutions. His famous formula for π can be derived with repeated use of trigonometric functions.

Up until this point, sine values, as well as the other trigonometric values, were still regarded as actual line lengths and not ratios of lengths. After the invention of calculus, Leonhard Euler (1707–83) came to realize that it is appropriate to think of sine not as a physical length, but rather as a function of angle independent of length. He suggested that this could be accomplished by scaling all circles under consideration to unit circles, an operation that is equivalent to dividing all quantities by the radius of the circle. Thus, for the first time, all trigonometric quantities came to be thought of as ratios. In 1748 Euler wrote "Introductio in analysin infinitorum", which became the dominating textbook on the topic of trigonometry for the century that followed. It essentially outlines the principles of trigonometry as we regard them today.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 123. Identify the individuals based on the descriptions from the text in Activity 122.

Abu al-Wafa / Aryabhata / Bartholomaeus Pitiscus / Bhaskara II / Claudius Ptolemy / Eratosthenes of Cyrene / Fibonacci / François Viète / Georg Joachim Rheticus / Hipparchus of Rhodes / Leonhard Euler / Regiomontanus / Thales of Miletus / Thomas Fincke

1. A Danish physician who coined the terms tangent and secant and developed further fundamental trigonometric formulae.
2. A famous scholar who travelled extensively in Arab countries and wrote about trigonometry in his work "Practica geometriae."
3. A French mathematician who showed how to solve certain cubic equations by making trigonometric substitutions.
4. A German astronomer and mathematician who compiled "De triangulis omnimodis," a compendium of trigonometry.
5. A German mathematician who invented the term "trigonometry."
6. A German mathematician who published "Canon doctrinae triangulorum," defining all six basic trigonometric functions.
7. A Greek philosopher credited with using similar triangles to determine the height of the Cheops pyramid.
8. A Greek scholar who computed the circumference of the Earth using lengths of shadows and a simple geometric argument on angles.
9. An Indian scholar who developed sophisticated techniques for computing half-chord values, contributing to trigonometry.
10. A mathematician of Baghdad who systemized theorems and proofs of Indian trigonometry, and possibly invented the tangent function.

11. A mathematician who constructed a table of chord lengths for a circle, contributing to the development of trigonometry.
12. A mathematician who wrote an extensive treatise on the theory of chords and their use in obtaining information about spherical triangles.
13. A mathematician who, after the invention of calculus, suggested thinking of sine as a function of angle independent of length.

Activity 124. Determine whether the statements are true or false by quoting from the text in Activity 122.

1. Ancient texts, like the Rhind papyrus and a Babylonian tablet, feature trigonometric problems, including finding pyramid slope angles and tabulating secant values.
2. Claudius Ptolemy's treatise centred on triangles created by straight lines on a sphere's surface.
3. Greek astronomers opted to use measures of straight lengths rather than working directly with angles in their models for celestial motion.
4. Hipparchus of Rhodes played a role in trigonometry by constructing a table of chord lengths for circles.
5. Indian scholars like Aryabhata and Bhaskara II simplified trigonometry in the 5th century by using half-chords, a concept similar to today's cosine function.
6. Thales of Miletus employed congruent triangles to determine the height of the Cheops pyramid by comparing its shadow length to that of a rod in the ground.
7. The Babylonian clay tablet from 1700 B.C.E. includes a table of tangent values for angles between 15° and 30°.
8. The German mathematician Regiomontanus compiled "De triangulis omnimodis," a trigonometry compendium in 1464.
9. The term "trigonometry" is attributed to German mathematician Bartholomaeus Pitiscus (1561–1613).
10. The use of triangles by surveyors, architects, navigators, and astronomers from ancient times contributed to the formation of trigonometry.

Activity 125. In groups, discuss the points. Refer to the text in Activity 122.

1. Explore the historical applications of trigonometry, such as its use by surveyors, architects, navigators, and astronomers. Discuss how these practical applications contributed to the development of trigonometry as a formal subject.
2. Examine the cultural contributions to trigonometry from different civilizations, including ancient Egypt, Babylon, Greece, India, and the Arab world. How did each culture contribute to the evolution of trigonometric concepts and techniques?

3. Trace the evolution of trigonometric functions from their early forms, such as chord lengths, to the more familiar sine and cosine functions. Discuss how different cultures and mathematicians shaped the understanding and representation of trigonometric concepts.
4. Investigate the role of trigonometry in ancient astronomy, as demonstrated by figures like Hipparchus, Ptolemy, and the Indian scholars. How did trigonometric methods contribute to the study of celestial bodies and the development of astronomical models?
5. Explore how mathematical knowledge, particularly trigonometry, was transmitted across cultures. Discuss the impact of translations, travels, and exchanges between different civilizations, such as the interactions between European, Arab, and Indian scholars.
6. Reflect on the dual nature of trigonometry, both as a practical tool for real-world problem-solving and as a theoretical pursuit in mathematics. Discuss examples of how trigonometry was applied in astronomy, surveying, and algebra, and how it contributed to the advancement of mathematical principles.
7. Investigate the conceptual shift from regarding trigonometric values as actual lengths to understanding them as ratios. Discuss the implications of this shift, especially in the context of Euler's insights and the development of calculus.
8. Focus on the contributions of key mathematicians like Fibonacci, Regiomontanus, Rheticus, and Euler to the development of trigonometry. How did their works shape the understanding of trigonometric functions, and what role did they play in advancing mathematical knowledge?
9. Explore the interdisciplinary connections between trigonometry and other branches of mathematics, such as algebra and calculus. Discuss how trigonometric concepts were integrated into solving cubic equations, deriving formulas like π, and contributing to the foundations of calculus.
10. Examine the enduring impact of Euler's "Introductio in analysin infinitorum" as a dominant textbook on trigonometry. Discuss how Euler's work influenced subsequent generations of mathematicians and shaped the principles of trigonometry as they are regarded today.

Activity 126. Choose one point in Activity 125 and elaborate on it in writing. Refer to the text in Activity 122.

“The fact that all mathematics is symbolic logic
is one of the greatest discoveries of our age.”

(Bertrand Russell)