“The fact that all mathematics is symbolic logic
is one of the greatest discoveries of our age.”

(Bertrand Russell)

Module 4. Areas of Advanced Mathematics

Unit 16. Set Theory

Activity 127. Complete the table with the names of the symbols.

a curly bracket / a round bracket / a square bracket		
Symbol	**British**	**American**
( )	a bracket (1) __________	a parenthesis (pl. parentheses)
[ ]	(2) __________	a bracket
{ }	(3) __________ a brace	

Activity 128. What symbols and signs are used in mathematics? Watch the video “Where Do Math Symbols Come From?” to choose the best answer to the questions. Then watch the video again and make a note of all the mathematical signs and symbols and their purpose. Extend the list with your examples.

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1. Why did Robert Recorde create the equal sign?
A. He wanted to make mathematics more difficult for students
B. He was tired of repeatedly writing "is equal to" in his work
C. He needed a symbol that looked like Greek letters
D. He wanted to replace all words in mathematics with symbols

2. Where does the plus sign for addition come from?
A. A shortened form of the Latin word "et"

B. Two parallel lines that represent equality
C. An exclamation mark used by Christian Kramp
D. A Greek letter that means "more"

3. What is the main reason mathematicians invented symbols?
A. To make mathematics impossible for non-mathematicians to understand
B. To communicate with alien civilizations
C. To avoid repetition and lengthy written explanations
D. To make their work look more impressive and complex

4. How is the capital sigma symbol described?
A. It represents a number that cannot be written in decimal form
B. It shows how many times to repeat a multiplication operation
C. It condenses a long string of sequential terms being added together
D. It indicates when to divide a result by three

5. What can be inferred about mathematical symbols?
A. All symbols were carefully chosen to visually represent their meaning
B. Symbols are a universal language that all civilizations would use identically
C. Understanding mathematical symbols requires memorization and practice, like learning a language
D. Mathematical symbols are more important than understanding the concepts behind them

Activity 129. Match the words with the definitions.

1. finite
2. infinite
3. infinity
4. set
5. subset
6. superset

a. a group of mathematical quantities that have some characteristic in common
b. a set consisting of elements of a given set that can be the same as the given set or smaller
c. a set consisting of elements of a given set that is larger than the given set
d. able to be put in a one-to-one correspondence with part of itself
e. capable of being completely counted
f. the concept of a value greater than any finite numerical value

Activity 130. Read the article. In pairs, discuss the questions in the box.

Loosely speaking, a set is any collection of objects or numbers specified in a well-defined manner. Each item in the set is called an element, or a member, of the set. For example, “dog” is an element of the set of mammals. If an entity “a” is an element of a set S, we write a ∈ S. If “a” does not belong to S, we write a ∉ S.

Sets are typically specified either by listing the elements of the set between a set of braces “{ }”, or listing a few elements of the set to indicate a pattern. For example {a, e, i, o, u} is the set consisting of the five vowels of the alphabet, and {3, 6, 9, 12, ...} is the set of all multiples of 3. It may also be possible to define a set as consisting of elements from some universal collection that satisfy a certain property. For example, {x ∈ R | x > 5} denotes the set of all real numbers that are greater than 5. (Some mathematicians prefer to use a colon “:” instead of a vertical bar in this notation.)

The order in which the elements of a set are listed is immaterial. For example, {A, 6, *} and {*, 6, A} are the same set. Also, elements of a set are listed without repetition. For instance, the set {a, a, a, a, a} is the set with a single element “a”. The empty set (the null set, the void set) is the set that contains no elements.

Two sets are deemed equal if they possess precisely the same elements. For example, the sets {2, 4, 6, 8, ...} and {n | n is a counting number divisible by 2} are equal sets. A set A is said to be a subset of a set B if every element of A is also a member of B. We write A ⊂ B if we are certain that the two sets are not equal, and A ⊇ B if equality of the sets is possible. For example, the set of all multiples of 4 is a subset of the set of all multiples of 2.

Although the intuitive notion of a set as a collection of objects is as ancient as the human race, the idea of a set as a formal mathematical concept was not proposed until the 19^{th} century. In his development of Boolean algebra, The British mathematician George Boole (1815–64) introduced the notion of set as a fundamental tool for the study of formal logic. The German mathematician Georg Cantor (1845–1918), in his attempts to understand the foundation of all of mathematics, came to regard sets as even more basic and fundamental than the notion of number. Cantor properly formalized a theory of set manipulations and introduced the striking notion of cardinality (the cardinality of a set is a measure of the number of elements of the set). His work led him to profound insights into the nature of finite and infinite sets, leading him to extend the concept of number to include more than one type of infinity.

Intuitively, a set is said to be finite if one can recite all the elements of the set in a bounded amount of time. For instance, the set {knife, fork, spoon} is finite, for it takes only a second or two to recite the elements of this set. On the other hand, the set of natural numbers {1, 2, 3, ...} is not finite, for one can never recite each and every element of this set.

Despite our intuitive understanding of the concept, it is difficult to give a precise and direct mathematical definition of a finite set. The easiest approach is to simply define a finite set to be one that is not infinite, since the notion of an infinite set can be made clear. Alternatively, since there is a well-defined procedure for mechanically writing down the string of natural numbers 1, 2, 3, ..., one can define a finite set to be any set S whose elements can be put in one-to-one correspondence with a bounded initial segment of the string of natural numbers. For instance, matching “knife” with 1, “fork” with 2, and “spoon” with 3, the set {knife, fork, spoon} is finite because its elements can be matched precisely with the string of natural numbers {1, 2, 3}.

In 1902 the British mathematician and philosopher Bertrand Arthur William Russell (1872–1970) stunned the mathematical community with his construction of a simple paradox, today called Russell’s paradox, that shows that our naive understanding of the notion of set is fundamentally flawed. Although Cantor believed that set theory is the foundation on which all of mathematics is built, it became clear to mathematicians that the concept of a set and what it means to be an “element of” must remain as undefined terms. In the decades that followed, mathematicians such as Ernst Friedrich Ferdinand Zermelo (1871–1953) attempted to develop an axiomatic theory of sets (based on undefined terms) that successfully avoids Russell’s paradox. To this day, not all mathematicians agree that this goal has yet been achieved.

(from Encyclopaedia Britannica)

1. What is a set?
2. How are sets typically specified?
3. What is the significance of the order in which elements are listed in a set?
4. What types of sets are there?
5. Who introduced the notion of sets?
6. How did George Boole use sets in the development of Boolean algebra?
7. What did Georg Cantor contribute to the understanding of sets and the concept of infinity?
8. How is the finiteness of a set intuitively described, and why is it challenging to provide a precise mathematical definition?

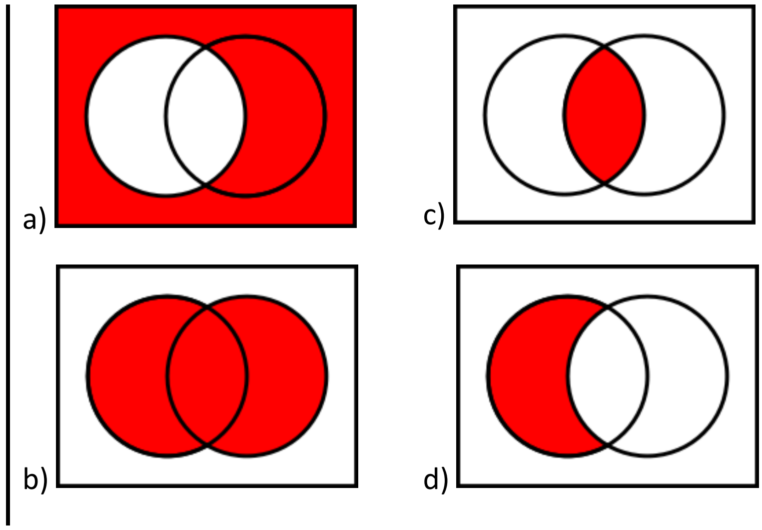
Activity 131. Reorder the sentences to make a text on set theory.

a. A finite set has a definite number of members; such a set might consist of all the integers from 1 to 1,000 or all marked bus stops along a given route.
b. A set is commonly represented by a list of its members, or elements, enclosed within braces; the statement that a set called A comprises the numbers 1, 2, and 3 is made by the expression A = {1, 2, 3}.
c. A set that has no members is called an empty set (or a null or void set) and is denoted by the symbol Ø.

d. An infinite set has an endless number of members; all the positive integers or all points along a given line compose infinite sets.
e. Set theory is a branch of mathematics that deals with the properties of well-defined collections of objects, which may be of a mathematical nature, such as numbers or functions, or not.
f. Sets may be finite or infinite.

Activity 132. Match the names of set operations with the Venn diagrams.

1) set complement
2) set difference
3) set intersection
4) set union



Activity 133. In writing, comment on the quote from the text in Activity 130, justifying the author's position as well as expressing your stance.

"In his attempts to understand the foundation of all of mathematics, the German mathematician Georg Cantor came to regard sets as more basic and fundamental than the concept of number."

Unit 17. Mathematical Logic

Activity 134. Complete the table with the types of mathematical statements.

axiom / conjecture / corollary / lemma / postulate / theorem	
Statement	**Example**
(1) ___________ (2) ___________ (Assumption)	0 is a natural number.
(3) ___________ (Hypothesis)	Every even natural number greater than 2 is the sum of two prime numbers.
(4) ___________ (Proposition)	The sum of the squares on the legs (catheti) of a right triangle is equal to the square on the hypotenuse (the side opposite the right angle).
(5) ___________	If a prime "p" divides the product "ab" of two integers "a" and "b", then "p" must divide at least one of those integers "a" or "b".
(6) ___________	All internal angles in a rectangle are right angles. All internal angles in a square are right angles.

Activity 135. Read the two paragraphs. Conclude which one explores deductive reasoning and which — inductive.

1. In the scientific method, there are two general processes for establishing results. The first, called ___________ reasoning, arrives at general conclusions by observing specific examples, identifying trends, and generalizing. "The sun has always risen in the past, therefore it will rise tomorrow," for example, illustrates this mode of reasoning. The ___________ process relies on discerning patterns but does not attempt to prove that the patterns observed apply to all cases. (Maybe the sun will not rise tomorrow.) For this reason, a conclusion drawn by the ___________ process is called a conjecture or an educated guess. If there is just one case for which the conclusion does not hold, then the conjecture is false. Such a case is called a counterexample.

2. On the other hand, ___________ reasoning works to prove a specific conclusion from one or more general statements using logical reasoning (as given by formal logic) and valid

arguments. For example, given the statements, “All cows eat grass” and “Daisy is a cow,” we can conclude, by __________ reasoning, that Daisy eats grass. __________ reasoning does not rely on the premises that are made necessarily being true. For example, “Sydney and Boston are planets, therefore Boston is a planet” is a valid argument, whereas “Either Boston or Venus is a planet, therefore Venus is a planet” is invalid.

Activity 136. Complete the sentences with the words from the text in Activity 137.

1. In mathematics, the systematic study of __________ is called formal logic or symbolic logic.
2. It analyzes the structure of arguments, as well as the methods and validity of mathematical deduction and __________.
3. He sought to identify modes of __________ that are valid by virtue of their structure, not their content.
4. This mode of thought allowed Euclid (ca. 300–260 B.C.E.) to formalize geometry, using deductive __________ to __________ geometric truths from a small collection of axioms (self-evident truths).
5. Propositional logic is the part which deals with __________ involving simple declarative sentences (statements) joined by connectives.
6. Beginning with an impressively minimal set of __________ (“self-evident” logical principles), they attempted to establish the logical foundations of all of mathematics.

Activity 137. Read the article. Determine whether the statements in the box below are true or false by quoting from the text.

In mathematics, the systematic study of reasoning is called formal logic or symbolic logic. It analyzes the structure of arguments, as well as the methods and validity of mathematical deduction and proof.

The principles of logic can be attributed to Aristotle (384–322 B.C.E.), who wrote the first systematic treatise on the subject. He sought to identify modes of inference that are valid by virtue of their structure, not their content. For example, “Green and blue are colours; therefore green is a colour” and “Cows and pigs are reptiles; therefore cows are reptiles” have the same structure (“A and B, therefore A”), and any argument made via this structure is logically valid. (In particular, the second example is logically sound.) This mode of thought allowed Euclid (ca. 300–260 B.C.E.) to formalize geometry, using deductive proofs to infer geometric truths from a small collection of axioms (self-evident truths).

No significant advance was made in the study of logic for the millennium that followed. This period was mostly a time of consolidation and transmission of the material from antiquity. The Renaissance, however, brought renewed interest in the topic. Mathematical scholars of the time, Pierre Hérigone and Johann Rahn in particular, developed means for representing logical arguments with abbreviations and symbols, rather than words and sentences. Gottfried Wilhelm Leibniz (1646–1716) came to regard logic as “universal mathematics.” He advocated the development of a “universal language” or a “universal calculus” to quantify the entire process of mathematical reasoning. He hoped to devise new mechanical symbolism that would reduce errors in thinking to the equivalent of arithmetical errors. (He later abandoned work on this project, assessing it too daunting a task for a single man.)

In the mid-1800s George Boole succeeded in creating a purely symbolic approach to propositional logic, that part which deals with inferences involving simple declarative sentences (statements) joined by the connectives: not, and, or, if ... then ..., iff (if and only if). (These are called the negation, conjunction, disjunction, conditional, and the biconditional, respectively.) He successfully applied it to mathematics, thereby making a significant step to achieving Leibniz’s goal.

In 1879 the German mathematician and philosopher Gottlob Frege constructed a symbolic system for predicate logic. This generalizes propositional logic by including quantifiers: statements using words such as “some”, “all”, and “no”. (For example, “All men are mortal” as opposed to “This man is mortal.”) At the turn of the century David Hilbert sought to devise a complete, consistent formulation of all of mathematics using a small collection of symbols with well-defined meanings. The English mathematician and philosopher Bertrand Russell, in collaboration with his colleague Alfred North Whitehead, took up Hilbert’s challenge. In 1925 they published a monumental work. Beginning with an impressively minimal set of premises (“self-evident” logical principles), they attempted to establish the logical foundations of all of mathematics. This was an impressive accomplishment. (After hundreds of pages of symbolic manipulations, they established the validity of “1 + 1 = 2,” for example.) Although they did not completely reach their goal, Russell and Whitehead’s work has been important for the development of logic and mathematics.

Six years after the publication of their efforts, however, Kurt Gödel stunned the mathematical community by proving Hilbert’s (and Leibniz’s) goal to be futile. He demonstrated once and for all that any formal system of logic sufficiently sophisticated to incorporate basic principles of arithmetic cannot attain all the statements it hopes to prove. His results are today called Gödel’s incompleteness theorems. The vision to reduce all truths of reason to incontestable arithmetic was thereby shattered.

(from Encyclopaedia Britannica)

1. Aristotle wrote the first treatise on logic, focusing on the content rather than the structure of arguments.
2. Bertrand Russell and Alfred North Whitehead's work in 1925 attempted to establish the logical foundations of mathematics.
3. David Hilbert aimed to formulate all of mathematics consistently, inspiring Russell and Whitehead's monumental 1925 work.
4. Formal logic, also known as symbolic logic, systematically studies reasoning in mathematics, analyzing argument structures and the validity of deductions and proofs.
5. George Boole's symbolic approach to propositional logic had no significant impact on mathematics.
6. In 1879, Gottlob Frege developed a symbolic system for predicate logic, extending propositional logic with quantifiers like "some" and "all."
7. Kurt Gödel's incompleteness theorems supported Hilbert's goal.
8. Leibniz regarded logic as "universal mathematics" and advocated for the development of a "universal language" in mathematical reasoning.
9. The Renaissance was a period of decline for logical studies.
10. The vision to reduce all truths of reason to incontestable arithmetic remains intact despite Gödel's results.

Activity 138. Reorder the sentences to make a text on foundations of mathematics.

a. He too searched for small collections of concepts that were fundamental and, hopefully, common to all fields.
b. In the late 1800s and at the turn of the century with the discovery of Russell's paradox in set theory, mathematicians were led to apparent paradoxes and inconsistencies within the seemingly very basic notions of "set" and "number."
c. Leonhard Euler (1707–83) produced fundamental results in disparate branches of mathematics and often saw connections between those branches.
d. The branch of mathematics concerned with the justification of mathematical rules, axioms, and modes of inference is called foundations of mathematics.
e. The paradigm for critical mathematical analysis came from the work of the great geometer Euclid (ca. 300–260 B.C.E.) who, in his work "The Elements", demonstrated that all geometry known at his time can be logically deduced from a small set of self-evident truths (axioms).
f. This led to the fervent study of the fundamental principles of elementary mathematics and even to the study of the process of mathematical thinking itself (formal logic).

g. Understanding the philosophical foundations of mathematics is still an area of intense scholarly research.

Activity 139. Are there unanswerable questions and unsolvable problems to mathematics? If so, give examples. Watch the video "The Paradox at the Heart of Mathematics: Gödel's Incompleteness Theorem" to choose the best answer to the questions. Describe how Gödel's discovery upended mathematics.

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1. What paradox inspired Kurt Gödel's mathematical discovery?
A. A statement that refers to itself and creates a logical contradiction
B. A complex equation that nobody could solve
C. An ancient Greek mathematical problem
D. A contradiction in basic arithmetic operations

2. How did Gödel make mathematics "talk about itself"?
A. He wrote mathematical statements in different languages
B. He created new axioms that described other axioms
C. He translated mathematical statements into code numbers
D. He used words instead of numbers in equations

3. According to Gödel's incompleteness theorem, what is true about axiomatic systems?
A. They can all be completed if enough axioms are added
B. They always contain true statements that cannot be proved
C. They only work for simple mathematical problems
D. They were proven wrong by modern computers

4. How did most mathematicians respond to Gödel's discovery?
A. They accepted the new reality it presented
B. They proved that his theorem was incorrect
C. They stopped working on mathematical proofs
D. They ignored it completely and continued as before

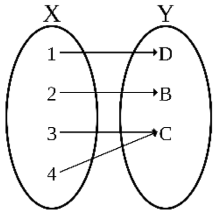
5. What positive outcome resulted from Gödel's theorem?
A. It allowed mathematicians to prove every mathematical claim
B. It eliminated all paradoxes from mathematics
C. It inspired important developments in early computers
D. It created more than 350 proofs of the Pythagorean theorem

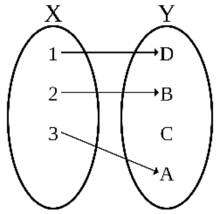
Activity 140. Single out the keywords (key phrases) from the text in Activity 137 and write an abstract of it.

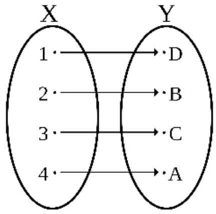
Unit 18. Function Theory

Activity 141. Match the functions with the diagrams.

1) bijection (bijective, one-to-one)
2) injection (injective, one-to-one)
3) surjection (surjective, onto)

a) 

b) 

c) 

Activity 142. Match the words with the definitions.

1. codomain
2. dependent variable
3. domain
4. graph
5. independent variable
6. range
7. value

a. a drawing or a visual representation that shows the relationship between two or more variables
b. a particular magnitude, number, or quantity
c. a variable in a functional relation whose value determines the value or values of other variables
d. a variable in a functional relation whose value is determined by the values assumed by other variables in the relation
e. the set of all possible outputs for a given function
f. the set of values of the independent variable of a function for which the functional value exists
g. the set of values that a function is allowed to take

Activity 143. Find the definitions of the term "function" in the paragraphs of Activity 144 proposed by the mathematicians.

1. Georg Cantor
2. Gottfried Wilhelm Leibniz
3. Jean Baptiste Joseph Fourier
4. Leonhard Euler
5. Nicole Oresme
6. Peter Gustav Lejeune Dirichlet

Activity 144. Reorder the paragraphs to make a text on function theory.

a. Advanced texts in mathematics today typically present all three definitions of a function — as a formula, as a set of ordered pairs, and as a mapping — and mathematicians will typically work with all three approaches.
b. In 1694 the German mathematician Gottfried Wilhelm Leibniz, codiscoverer of calculus, coined the term “function” to mean the slope of the curve, a definition that has very little in common with our current use of the word. The great Swiss mathematician Leonhard Euler (1707–83) recognized the need to make the notion of a relationship between quantities explicit, and he defined the term “function” to mean a variable quantity that is dependent upon another quantity. Euler introduced the notation *f*(x) for “a function of x,” and promoted the idea of a function as a formula. He based all his work in calculus and analysis on this idea, which paved the way for mathematicians to view trigonometric quantities and logarithms as functions. This notion of function subsequently unified many branches of mathematics and physics.
c. In 1822 the French physicist and mathematician Jean Baptiste Joseph Fourier presented work on heat flow. He represented functions as sums of sine and cosine functions but commented that such representations may be valid only for a certain range of values. This later led the German mathematician Peter Gustav Lejeune Dirichlet (1805–1859) to propose a more precise definition: A function is a correspondence that assigns a unique value of a dependent variable to every permitted value of an independent variable. This, on an elementary level, is the definition generally accepted today.
d. In the late 19^{th} century, the German mathematician Georg Cantor (1845–1918) attempted to base all of mathematics on the fundamental concept of a set. Because the terms variable and relationship are difficult to specify, Cantor proposed an alternative definition of a function: A function is a set of ordered pairs in which every first element is different.
e. In the mid-1300s the French mathematician Nicole Oresme discovered that a uniformly varying quantity (such as the position of an object moving with uniform

velocity, for instance) could be represented pictorially as a graph, and that the area under the graph represents the total change of the quantity. Oresme was the first to describe a way of graphing the relationship between an independent variable and a dependent one and, moreover, demonstrate the usefulness of the task.

f. Mathematicians consequently came to think of functions as “mappings” that assign to elements of one set X, called the domain of the function, elements of another set Y, called the codomain. (Each element “x” of X is assigned just one element of Y.) One can thus depict a function as a diagram of arrows in which an arrow is drawn from each member of the domain to its corresponding member of the codomain. The function is then the complete collection of all these correspondences.

g. Since the time of antiquity, scholars were interested in identifying rules or relationships between quantities. For example, the ancient Egyptians were aware that the circumference of a circle is related to its diameter via a fixed ratio that we now call “pi”, and Chinese scholars, and later the Pythagoreans, knew that the three sides of a right triangle satisfy the simple relationship given by Pythagoras’s theorem. Although these results were not expressed in terms of formulae and symbols (the evolution of algebraic symbolism took many centuries), scholars were aware that the value of one quantity could depend on the value of other quantities under consideration. Although not explicit, the notion of a “function” was in mind.

h. This idea is based on the fact that the graph of a function is nothing more than a collection of points (x,y) with no two y-values assigned to the same x-value. Cantor’s definition is very general and can be applied not only to numbers but to sets of other things as well.

Activity 145. Single out the keywords (key phrases) from the text in Activity 144 and write an abstract of it.

Activity 146. In groups, discuss the points. Refer to the text in Activity 144.

1. Discuss how the understanding of mathematical concepts, such as the notion of a "function," has evolved over time, from ancient civilizations to the modern era. What role did different cultures and mathematicians play in shaping these concepts?
2. Explore the significance of Nicole Oresme's discovery in the mid-1300s regarding the representation of a uniformly varying quantity as a graph. How did this visual representation contribute to the understanding of relationships between independent and dependent variables?
3. Analyze the diverse definitions of a "function" proposed by Leibniz, Euler, Dirichlet, and Cantor. How did these definitions reflect the mathematical thinking and

challenges of their respective time periods? In what ways did these definitions contribute to the unification of mathematical branches?

4. Examine the role of notation, specifically Euler's introduction of *f*(x) for "a function of x," in formalizing mathematical concepts. How did the use of symbols and formulae contribute to the development of calculus and analysis?
5. Delve into Georg Cantor's attempt to base all of mathematics on the concept of a set and his alternative definition of a function as a set of ordered pairs. How did this set-based perspective impact the understanding of functions?
6. Discuss the modern conceptualization of functions as "mappings" between sets, with a domain and a codomain. How does this perspective offer a unified understanding of functions, and what advantages does it provide in contemporary mathematics?
7. Consider the approach of presenting all three definitions of a function — as a formula, as a set of ordered pairs, and as a mapping — in advanced mathematics texts. How does this integrated approach enhance the comprehension and application of mathematical concepts?
8. Explore the practical implications of the historical developments in mathematical understanding, particularly in terms of how mathematicians work with and apply the concept of a function in various fields and disciplines.

Activity 147. Choose one point in Activity 146 and elaborate on it in writing. Refer to the text in Activity 144.

Unit 19. Mathematical Analysis and Calculus

Activity 148. Read the passage. In pairs, discuss the questions in the box.

Any topic in mathematics that makes use of the notion of a limit in its study is called analysis. Calculus comes under this heading, as does the summation of infinite series, and the study of real numbers. The Greek mathematician Pappus of Alexandria (ca. 320 C.E.) called the process of discovering a proof or a solution to a problem "analysis." He wrote about "a method of analysis" somewhat vaguely in his geometry text "Collection", which left mathematicians centuries later wondering whether there was a secret method hidden behind all of Greek geometry. The great René Descartes (1596–1650) developed a powerful method of using algebra to solve geometric problems. His approach became known as analytic geometry. The branch of mathematics that deals with the notion of continuous growth and change is called calculus (infinitesimal calculus). It is based on the concept of infinitesimals, exceedingly small quantities, and on the concept of a limit, quantities that can be approached more and more closely but never reached. The branch of calculus known as differential calculus deals with notions of slope, rates of change and ratios of change. For example, a study of velocity, which can be described as the rate of change of position, falls under the study of differential calculus, as do other concepts that arise in the study of motion. Any process that involves segmenting a quantity into manageable pieces, summing, and taking the limit of these sums as the process is refined falls under the category of integral calculus. The word "calculus" comes from the Latin word "calx" for "pebble," which in turn is derived from the Greek word "chalis" for "limestone." Small beads or stones arranged in a counting board or on an abacus were often used to aid mathematical calculations, and the word "calculus" came to refer to all mathematical activity. Today, however, the word is used almost exclusively to denote the study of continuous change.

(from Encyclopaedia Britannica)

1. Define the terms "limit" and "infinitesimal".
2. What is the origin of the word "calculus"?
3. What is the difference between analysis and calculus?
4. How has the use of the words "analysis" and "calculus" evolved?
5. Distinguish between infinitesimal calculus, differential calculus, and integral calculus.

Activity 149. Do the quiz on calculus. In pairs, compare your answers.

1. **When did the study of calculus begin?**
    a. ancient times
    b. the 17th century
    c. the 18th century
    d. the 19th century
2. **Who among the ancient Greek scholars is mentioned for their work on infinitesimals?**
    a. Plato
    b. Pythagoras
    c. Archimedes
    d. Eudoxus
3. **What method was developed by Greek scholars to compute the area or volume of a curved figure?**
    a. method of indivisibles
    b. method of limits
    c. method of exhaustion
    d. method of infinitesimals
4. **Which mathematician wrote the first textbook on integration methods in 1635?**
    a. Johannes Kepler
    b. Bonaventura Cavalieri
    c. Pierre de Fermat
    d. John Wallis
5. **In the mid-1600s, what breakthrough united the study of tangent problems and area problems?**
    a. inverse relationship discovery
    b. fundamental theorem of calculus
    c. method of exhaustion
    d. Kepler's optimization solutions
6. **How did Newton refer to the quantity being studied in calculus and its rate of change?**
    a. fluent and fluxion
    b. element and derivative
    c. variable and gradient
    d. integral and differential
7. **Who developed a notational system for calculus accessible to a wide audience in the 17th century?**

a. Isaac Newton
b. Pierre de Fermat
c. Blaise Pascal
d. Gottfried Wilhelm Leibniz

8. Which 18th-century mathematician questioned the validity of calculus in his scathing essay "The Analyst"?

a. Leonhard Euler
b. George Berkeley
c. Karl Weierstrass
d. Augustine Louis Cauchy

9. Who is credited with developing a concept of integration applied to a wider class of functions in the 19th century?

a. Bernhard Riemann
b. Joseph-Louis Lagrange
c. Henri Léon Lebesgue
d. Pierre-Simon Laplace

10. What idea did Augustine Louis Cauchy propose to replace the notion of an infinitesimal?

a. method of limits
b. method of exhaustion
c. measure theory
d. fluxion theory

Activity 150. Read the article. Review your answers to the quiz in Activity 149.

The study of calculus begins with the study of motion, a topic that has fascinated and befuddled scholars since the time of antiquity. The first recorded work of note in this direction dates back to the Greek scholars Pythagoras (ca. 569–475 B.C.E) and Zeno of Elea (ca. 500 B.C.E.), and their followers, who put forward the notion of an infinitesimal as one possible means for explaining the nature of physical change. Motion could thus possibly be understood as the aggregate effect of a collection of infinitely small changes. Zeno, however, was very much aware of fundamental difficulties with this approach and its assumption that space and time are consequently each continuous and thus infinitely divisible. Through a series of ingenious logical arguments, Zeno reasoned that this cannot be the case. At the same time, Zeno presented convincing reasoning to show that the reverse position, that space is composed of fundamental indivisible units, also cannot hold. The contradictory issues proposed by Zeno were not properly resolved for well over two millennia.

The concept of the infinitesimal also arose in the ancient Greek study of area and volume. Scholars of the schools of Plato (428–348 B.C.E.) and of Eudoxus of Cnidus (ca. 370 B.C.E.) developed a “method of exhaustion,” which attempted to compute the area or volume of a curved figure by confining it between two known quantities, both of which can be made to resemble the desired object with any prescribed degree of accuracy. Archimedes of Syracuse (287–212 B.C.E.) applied this method to compute the area of a section of a parabola, and 600 years later, Pappus of Alexandria (ca. 300–350 C.E.) computed the volume of a solid of revolution via this technique. Although successful in computing the areas and volumes of a select collection of geometric objects, scholars had no general techniques that allowed for the development of a general theory of area and volume. Each individual calculation for a single specific example was hailed as a great achievement in its own right.

The resurgence of scientific investigation in the mid-1600s led European scholars to push the method of exhaustion beyond the point where Archimedes and Pappus had left it. Johannes Kepler (1571–1630) extended the use of infinitesimals to solve optimization problems. Others worked on the problem of finding tangents to curves, an important practical problem, and the problem of finding areas of irregular figures. In 1635, the Italian mathematician Bonaventura Cavalieri wrote the first textbook on what we would call “integration methods”. He described a general “method of indivisibles” useful for computing volumes. The principle today is called Cavalieri’s principle.

The French mathematician Gilles Personne de Roberval (1602–75) was the first to link the study of motion to geometry. He realized that the tangent line to a geometric curve could be interpreted as the instantaneous direction of motion of a point travelling along that curve. The philosopher and mathematician René Descartes (1596–1650) developed general techniques for finding the formula for the tangent line to a curve at a given point. This technique was later picked up by Pierre de Fermat (1601–65), who used the study of tangents to solve maxima and minima problems in much the same way we solve such problems today. As a separate area of study, Fermat also developed techniques of integral calculus to find areas between curves and lengths of arcs of curves, which were later developed further by Blaise Pascal (1623–62) and English mathematicians John Wallis (1616–1703) and Isaac Barrow (1630–77).

At the same time scholars, including Wallis, began studying series and infinite products. The Scottish mathematician James Gregory (1638–75) developed techniques for expressing trigonometric functions as infinite sums, thereby discovering Taylor series 40 years before Brook Taylor (1685–1731) independently developed the same results.

By the mid-1600s, certainly, all the pieces of calculus were in place. Yet scholars at the time did not realize that all the varied problems being studied belonged to one unified whole, namely, that the techniques used to solve tangent problems could be used to solve area problems, and vice versa. A fundamental breakthrough came in the 1670s when, independently, Gottfried Wilhelm Leibniz (1646–1716) of Germany and Sir Isaac Newton

(1642–1727) of England discovered an inverse relationship between the "tangent problem" and the "area problem." The discovery of the fundamental theorem of calculus brought together the disparate topics being studied, provided a beautiful and natural perspective on the subject as a whole, and allowed scholars to make significant advances in solving geometric and physical problems with spectacular success. Despite the content of knowledge that had been established up until that time, it is the discovery of the fundamental theorem of calculus that represents the discovery of calculus.

Newton approached calculus through a concept of "flowing entities." He called any quantity being studied a "fluent" and its rate of change a "fluxion". Records show that he had developed these ideas as early as 1665, but he did not publish an account of his theory until 1704. Unfortunately, his writing style and choice of notation also made his version of calculus accessible only to a select audience. Leibniz, on the other hand, made explicit use of an infinitesimal in his development of the theory. He called the infinitesimal change of a quantity "x" a differential, denoted "dx". Leibniz invented a beautiful notational system for the subject that made reading and working with his account of the theory immediately accessible to a wide audience. (Many of the symbols we use today in differential and integral calculus are due to Leibniz.) Leibniz formulated his approach in the mid-1670s and published his account of the subject in 1684. Although it is now known that Newton and Leibniz had made their discoveries independently, matters at the time were not clear, and a bitter dispute arose over the priority for the discovery of calculus.

Applying the techniques to problems of the real world became the main theme of 18th-century mathematics. Newton's famous 1687 text "Principia" paved the way with its analysis of the laws of motion and the mechanics of the solar system. The Swiss brothers Jakob Bernoulli (1654–1705) and Johann Bernoulli (1667–1748) of the famous Bernoulli family, champions of Leibniz in the famous dispute, studied the newly invented calculus and were the first to give public lectures on the topic. Johann Bernoulli was hired to teach differential calculus to the French nobleman Guillaume François de L'Hopital (1661–1704) via written correspondence. In 1696 L'Hopital then published the content of Johann's letters with his own name as author. The Italian mathematician Marie Gaetana Agnesi (1718–99) wrote the first comprehensive textbook dealing with both differential and integral calculus in 1755.

The Swiss mathematician Leonhard Euler (1707–83) and French mathematicians Joseph-Louis Lagrange (1736–1813) and Pierre-Simon Laplace (1749–1827) were prominent in developing the theory of differential equations. Euler also wrote extensively on the subject of calculus, showing how the theory can be applied to a vast range of pure and applied mathematical problems. Yet despite the evident success of calculus, some 18th-century scholars questioned the validity and the soundness of the subject.

The sharpest critic of Newton's and Leibniz's work was the Anglican Bishop of Coyne, George Berkeley (1685–1753). In his scathing essay, "The Analyst," Berkeley demonstrated, convincingly, that both Newton's notion of a fluxion and Leibniz's concept of an infinitesimal are ill-defined, and that the foundations of the subject are consequently insecure. Mathematicians consequently began looking for ways to put calculus on a sound footing. Significant progress was not made until the 19^{th} century, when the French mathematician Augustine Louis Cauchy (1789–1857) suggested that the notion of an infinitesimal should be replaced by that of a limit. German mathematician Karl Weierstrass (1815–97) developed this idea further and was the first to give absolutely clear and precise definitions to all concepts used in calculus, devoid of any mystery or reliance on geometric intuition. The work of the German mathematician Richard Dedekind (1831–1916) highlighted the role properties of the real number system play in ensuring the validity of the intermediate-value theorem and extreme-value theorem and all the essential results that follow from them.

Initially, calculus was deemed a theory pertaining only to continuous change and continuous functions. The German mathematician Bernhard Riemann (1826–66) was the first to consider, and give careful discussion on, the integration of discontinuous functions. His definition of an integral is the one typically presented in textbooks today. At the end of the 19^{th} century, the French mathematician Henri Léon Lebesgue (1875–1941) literally turned Riemann’s approach around and developed a concept of integration that can be applied to a much wider class of functions and class of settings. In order to do this, Lebesgue had to develop a general “measure theory” for determining the size of complicated sets. His new theory proved to be fundamentally important, and it now has profound applications to a wide range of mathematical topics. It proved to be especially important to the sound development of probability theory.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 151. Identify the individuals based on the descriptions from the text in Activity 150.

Archimedes of Syracuse / Augustine Louis Cauchy / Bernhard Riemann / Blaise Pascal / Bonaventura Cavalieri / Eudoxus of Cnidus / George Berkeley / Gilles Personne de Roberval / Gottfried Wilhelm Leibniz / Guillaume François de L’Hopital / Henri Léon Lebesgue / Isaac Barrow / Jakob Bernoulli / James Gregory / Johann Bernoulli / Johannes Kepler / John Wallis / Joseph-Louis Lagrange / Karl Weierstrass / Leonhard Euler / Marie Gaetana Agnesi / Pappus of Alexandria / Pierre de Fermat / Pierre-Simon Laplace / Plato / Pythagoras / René Descartes / Richard Dedekind / Sir Isaac Newton / Zeno of Elea

1. The Anglican Bishop of Coyne, a critic of Newton's and Leibniz's work on calculus.

2. English mathematician who contributed to the study of tangents and integral calculus.
3. English mathematician who, independently of Leibniz, discovered the fundamental theorem of calculus.
4. Extended the use of infinitesimals to solve optimization problems in the mid-1600s.
5. French mathematician known for his work on celestial mechanics and contributions to calculus.
6. French mathematician who contributed to the theory of differential equations.
7. French mathematician who developed a concept of integration applied to a wider class of functions, introducing measure theory.
8. French mathematician who linked the study of motion to geometry.
9. French mathematician who suggested replacing infinitesimals with limits in calculus.
10. French nobleman who learned calculus from Johann Bernoulli and published the content of Johann’s letters.
11. German mathematician who considered the integration of discontinuous functions.
12. German mathematician who provided clear and precise definitions for calculus concepts.
13. German mathematician who, independently of Newton, discovered an inverse relationship between the "tangent problem" and the "area problem."
14. German mathematician whose work highlighted the role of the real number system in calculus.
15. Greek mathematician who applied the "method of exhaustion" to compute the area of a section of a parabola.
16. Greek philosopher associated with the school of thought studying area and volume.
17. Greek scholar known for his ingenious logical arguments regarding the difficulties of the infinitesimal approach and the nature of space and time.
18. Greek scholar linked to the development of the "method of exhaustion" for computing areas and volumes.
19. Greek scholar who, along with Zeno of Elea, contributed to the notion of an infinitesimal in explaining physical change.
20. Italian mathematician who wrote the first comprehensive textbook dealing with both differential and integral calculus.
21. Italian mathematician who wrote the first textbook on "integration methods" and introduced the "method of indivisibles."
22. Mathematician who computed the volume of a solid of revolution using the "method of exhaustion."
23. Mathematician who further developed integral calculus techniques.
24. Mathematician who used the study of tangents to solve maxima and minima problems and developed integral calculus techniques.
25. Philosopher and mathematician who developed general techniques for finding the formula for the tangent line to a curve.

26. Scottish mathematician who developed techniques for expressing trigonometric functions as infinite sums.
27. Swiss mathematician and brother of Jakob Bernoulli, who studied calculus and taught differential calculus to L'Hopital.
28. Swiss mathematician prominent in developing the theory of differential equations and extensively wrote on calculus.
29. Swiss mathematician who, along with Johann Bernoulli, studied and lectured on calculus.

Activity 152. Rearrange the events in chronological order according to the text of Activity 150. Provide dates where possible.

a. Bishop Berkeley questions the foundations of calculus.
b. Bonaventura Cavalieri introduces "method of indivisibles."
c. Cauchy suggests replacing infinitesimals with limits.
d. Cavalieri writes the first textbook on integration methods.
e. Dedekind emphasizes the role of the real number system in calculus.
f. Descartes, Fermat, Wallis, and Barrow contribute to tangent problems and areas of curves.
g. Greek scholars Pythagoras and Zeno of Elea propose the notion of an infinitesimal to explain physical change.
h. Greek scholars, including Plato, Eudoxus, and Archimedes, develop the "method of exhaustion" for computing areas and volumes.
i. Johannes Kepler extends infinitesimals to solve optimization problems.
j. Lebesgue develops a new concept of integration, introducing measure theory.
k. Lebesgue's theory profoundly impacts probability theory.
l. Leibniz and Newton independently discover the fundamental theorem of calculus, unifying tangent and area problems.
m. Newton's "Principia" analyzes laws of motion and solar system mechanics.
n. Riemann considers integration of discontinuous functions.
o. Roberval links the study of motion to geometry.
p. The Bernoulli brothers, L'Hopital, and Agnesi contribute to the application and teaching of calculus.
q. Weierstrass provides clear and precise definitions for calculus concepts.

Activity 153. Determine whether the statements are true or false by quoting from the text in Activity 150.

1. Calculus starts by examining the concept of infinity.
2. In the 1650s, Leibniz and Newton independently found a crucial connection between the "tangent problem" and the "area problem" in calculus.
3. Infinitesimals were employed by Johannes Kepler for solving optimization problems.
4. Initially, calculus was considered a theory unrelated to continuous change and continuous functions.
5. Leonhard Euler played a significant role in advancing the theory of differential equations.
6. Newton's approach to calculus involved the idea of "flowing entities."
7. Scholars possessed comprehensive techniques enabling the development of a general theory of area and volume.
8. The Anglican Bishop of Coyne, George Berkeley, supported Newton's and Leibniz's work.
9. The resurgence of scientific investigation influenced European scholars to extend the method of exhaustion beyond the point left by Archimedes and Pappus.
10. The Swiss brothers Jakob Bernoulli and Johann Bernoulli opposed Leibniz in the famous dispute.

Activity 154. In groups, discuss the points. Refer to the text in Activity 150.

1. How did the study of calculus evolve from the ancient Greeks' exploration of infinitesimals to the development of integral calculus and the fundamental theorem of calculus in the 17^{th} century?
2. Explore the "method of exhaustion" employed by Greek scholars and its application in computing areas and volumes. How did Archimedes and Pappus contribute to this method?
3. Discuss Cavalieri's principle as an early method of computing volumes and its significance in the development of calculus. How did scholars extend and refine these methods in the 17^{th} century?
4. Analyze the contributions of Gilles Personne de Roberval, René Descartes, and Pierre de Fermat in linking the study of motion to geometry. How did these developments lay the groundwork for calculus?
5. Investigate the 17^{th}-century scholars' exploration of series and infinite products, particularly the contributions of James Gregory. How did this contribute to the overall understanding of calculus?
6. Discuss the realization in the 17^{th} century that the techniques used to solve tangent problems could also be applied to solve area problems, leading to the fundamental theorem of calculus. How did this unify the disparate topics in calculus?

7. Explore the Newton-Leibniz dispute over the priority of the discovery of calculus. How did Newton's concept of "fluxions" compare to Leibniz's use of infinitesimals, and how did this controversy impact the development of calculus?
8. Investigate how 18th-century mathematicians, including Jakob and Johann Bernoulli, applied calculus to real-world problems, as seen in Newton's "Principia" and the public lectures on calculus.
9. Examine the challenges and criticisms posed by scholars like George Berkeley in the 18th century. How did these critiques prompt a search for a more rigorous foundation for calculus?
10. Explore the 19th-century developments by mathematicians like Cauchy, Weierstrass, and Dedekind in putting calculus on a sound footing. How did the introduction of limit concepts and precise definitions contribute to the development of calculus?

Activity 155. Choose one point in Activity 154 and elaborate on it in writing. Refer to the text in Activity 150.

Unit 20. Probability Theory and Mathematical Statistics

Activity 156. Match the words with the definitions.

1. census
2. event
3. expected value
4. inference
5. life table
6. outcome
7. probability
8. sample space

a. a complete enumeration of a population, typically including details about individuals' characteristics
b. a measure of the likelihood of an event occurring, expressed as a number between 0 and 1
c. a possible result of an experiment or random process
d. a subset of the sample space, representing a particular outcome or set of outcomes
e. a table showing the probability of survival and mortality at different ages in a population
f. the average value of a random variable, calculated as the sum of all possible values multiplied by their respective probabilities
g. the process of drawing conclusions about a population based on a sample of data from that population
h. the set of all possible outcomes of an experiment

Activity 157. What are the Wright brothers, Orville and Wilbur, credited with? Watch the video "The Coin Flip Conundrum" to choose the best answer to the questions. Describe the role that probability theory played in the brothers' business.

https://disk.yandex.ru/i/DjFvoSWGHuR5UQ

1. How did the Wright brothers originally decide who would fly their airplane first?
A. They had a complicated contest
B. They flipped a coin once
C. Wilbur automatically won the right
D. They flipped coins repeatedly

2. Why does the heads-heads combination take longer to achieve than heads-tails?
A. Because heads appears less frequently than tails
B. Because it requires more than two flips

C. Because it has a move that sends you back to the start
D. Because the probability of heads is lower

3. What example is used to explain why one combination takes longer than the other?
A. A board game comparison with different paths
B. Historical records from the Wright brothers
C. Scientific experiments with special coins
D. Computer simulations of coin flipping

4. According to the mathematical calculations, what is true about the average number of flips needed?
A. Both combinations require the same number of flips
B. Heads-heads requires fewer flips than heads-tails
C. Heads-tails requires fewer flips than heads-heads
D. The number of flips depends on who is flipping

5. What happened when the Wright brothers actually flipped the coin?
A. Orville won the flip and successfully flew the airplane
B. Wilbur won the flip but his flight attempt failed
C. They decided to flip multiple times instead
D. They used the heads-heads method to decide

Activity 158. Reorder the sentences to make a text on mathematical statistics.

a. Another statistic would be the tallest height recorded or the range of heights observed.
b. For example, a medical study might record the heights of 100 children, all age 8.
c. In leisure, many sports fans follow statistical analyses to assess team and player performance.
d. Insurance companies analyze life tables to make inferences and to set insurance rates.
e. It is based on the Latin verb “stare” meaning “to stand.”
f. Making a judgment based on the data that another child outside of the study is of abnormal height would be an example of using data for inferential purposes.
g. Statistics is an indispensable tool used in practically every aspect of life today.
h. Statistics is the branch of mathematics concerned with the methods of collecting, tabulating, and summarizing numerical facts (this is called descriptive statistics), and for making inferences and predictions based on these facts (inferential statistics).
i. Statistics is used extensively in government, business, and commerce to analyze opinion polls, campaign and advertising strategies, business operations, pollution

control, and other environmental concerns, for example, and as well as in scientific research and economic, political, and sociological studies.

j. The average height of the children would be an example of a statistic.

k. The numerical information gathered is called data, and an individual numerical fact about the data is called a statistic.

l. The word “statistic” was coined by the German political scientist Gottfried Achenwall (1719–72) to mean “a summary of how things stand.”

m. Weather predictions are based on methods of statistical inference, for example, as are the assessed effectiveness of new drugs, new medical procedures, and other health practices.

Activity 159. Do the quiz on probability theory and mathematical statistics. In pairs, compare your answers.

1. **What prompted the development of probability theory in the 17th century?**
   a. architectural calculations
   b. financial markets
   c. betting and gaming
   d. astrological predictions
2. **Who sought advice from Blaise Pascal about divvying up stakes in interrupted games, leading to the birth of probability theory?**
   a. Isaac Newton
   b. Chevalier de Méré
   c. Pierre de Fermat
   d. Jacob Bernoulli
3. **What is the key principle behind probability theory discussed by Girolamo Cardano and later recognized by Pascal and Fermat?**
   a. law of large numbers
   b. principle of equal likelihood
   c. expected value principle
   d. law of total probability
4. **Which mathematician recognized the wide-ranging applicability of probability beyond gambling, demonstrating its use in medicine and meteorology?**
   a. Pierre-Simon Laplace
   b. Jacob Bernoulli
   c. Carl Friedrich Gauss
   d. John Graunt

5. **What do probability and statistics explore?**
    a. Probability explores unknown collections; statistics explores known samples.
    b. Probability explores known samples; statistics explores unknown collections.
    c. Both explore unknown collections.
    d. Both explore known samples.
6. **Who recognized the repeated appearance of the normal distribution and wrote down a mathematical equation for it in 1733?**
    a. Jacob Bernoulli
    b. Abraham de Moivre
    c. Pierre-Simon Laplace
    d. Siméon Denis Poisson
7. **Which mathematician is considered the most important statistician of the 20^{th} century and transformed statistics into a powerful scientific tool?**
    a. Karl Pearson
    b. John von Neumann
    c. Ronald Aylmer Fisher
    d. William Sealy Gosset
8. **What significant statistical work did Francis Galton contribute in the 1860s?**
    a. inference from birth and death records
    b. analysis of human heredity
    c. development of chi-squared test
    d. introduction of hypothesis testing
9. **Who founded game theory in 1926, recognizing its applications to economics and social sciences?**
    a. Ronald Aylmer Fisher
    b. John Forbes Nash, Jr.
    c. William Sealy Gosset
    d. John von Neumann

Activity 160. Read the article. Review your answers to the quiz in Activity 159.

Questions in betting and gaming provided much of the early impetus for the development of probability theory. In 1654 Chevalier de Méré, a French nobleman with a taste for gambling, wrote a letter to the mathematician Blaise Pascal (1623–62) seeking advice about divvying up stakes from interrupted games.

For example, suppose, in a friendly game of tennis, two players each lay down a stake of $100 in a gamble to win “best out of nine” games, but rain interrupts play after just four games, with one player having won three games, the second only one. What then would be the fair way to divide the $200 pot? Of course, the division of money should somehow reflect each player's likelihood of winning the gamble if the series of games were to be finished.

Pascal communicated the concern of analyzing situations like these to his colleague Pierre de Fermat (1601–65), and their subsequent correspondences on the issue represented the birth of the new field of probability theory. Both mathematicians solved de Méré's "problem of points" (using two entirely different approaches, incidentally) and then later worked together to generalize the problem and extend their analyses to other types of games of chance. Their discoveries aroused the interest of other European scholars. In 1656 the Dutch physicist-astronomer-mathematician Christiaan Huygens (1629–95) published "On Reasoning in Games of Chance" summarizing and extending the ideas developed by Pascal and Fermat. He phrased their work in terms of a new notion, that of expected value. It proved to be very fruitful.

The key principle behind probability theory is the idea that if a situation can be described in terms of possible outcomes that are equally likely, then the probability of any particular outcome occurring is 1 divided by the total number of outcomes. This principle was actually first recognized and discussed more than a century earlier by the Italian mathematician and physician Girolamo Cardano (1501–76) in his work "Book on Games of Chance". This text, however, was not published until 1663, 9 years after Pascal and Fermat had solved de Méré's problem. It is likely that Cardano would be known as "the father of probability theory" had the work been published during his lifetime. Cardano also recognized the law of large numbers.

The Swiss mathematician Jacob Bernoulli (1654–1705) of the famous Bernoulli family recognized the wide-ranging applicability of probability in fields outside of gambling. His book "The Art of Conjecture", published posthumously in 1713, demonstrated the use of the theory in medicine and meteorology. It was also the first comprehensive text dealing with issues of statistics.

In some sense, probability and statistics represent two sides of the same fundamental situation. Probability explores what can be said about an unknown sample of a known collection. (For example, we know all possible numerical combinations from a pair of dice. What then is the most likely outcome from tossing a pair of dice?) Statistics explores what can be said about an unknown collection given a small sample. (If 37 of these 100 people brush their teeth twice a day, what can be said about teeth-brushing habits of the entire population?) The two fields remained closely intertwined during much of the 18th century and the early part of the next century.

In 1733 Abraham de Moivre (1667–1754) recognized the repeated appearance of the normal distribution in scientific studies and wrote down a mathematical equation for it. It first became apparent from the "randomness" of errors in astronomical observations and in scientific experiments.

The latter half of the 19^{th} century saw significant progress in developing and understanding the theoretical foundations of probability theory. This was chiefly due to the work of French mathematicians-astronomers-physicists Joseph-Louis Lagrange (1736–1813) and Pierre-Simon Laplace (1749–1827), German genius Carl Friedrich Gauss (1777–1855), and the French mathematician Siméon Denis Poisson (1781–1840) who, among other things, mathematically proved the law of large numbers. The most important publication in this era on the theory of probability was Laplace's 1812 text "Analytical Theory of Probability". In it, Laplace collected and extended everything known on the subject at that time. Russian mathematicians Pafnuty Chebyshev (1821–94), Andrei Markov (1856–1922), and Alexandr Lyapunov (1857–1918) further developed the mathematical underpinnings of the subject in the late 19^{th} century.

Basic statistical thought can be deemed as having developed considerably earlier. The ancient Egyptians compiled data concerning population and wealth as early as 3050 B.C.E., developing simple techniques to collate and record the numerical information gathered. The ancient Chinese undertook similar studies around 2300 B.C.E. A census was taken in 594 B.C.E. by the Greeks for the purpose of levying taxes, and Athens undertook a population census in 309 B.C.E. The Romans also kept census records, as well as records of births and deaths, and gathered significant quantities of numerical information from geographic surveys taken across the entire empire. Very few statistical records were kept during the period of the Middle Ages, however.

In 1662 John Graunt analyzed birth and death records and produced the first life table. The purpose of the table was to make general observations and predictions about life expectancy for classes of members of a particular population. This work represented a significant step toward analyzing data for the purposes of inference.

In 1790 the United States took its first decennial census, heralding the return of census taking. Several European nations followed suit soon afterward. The Belgian scholar Lambert Adolphe Quételet (1796–1874) analyzed the nation's records and made important observations about the influence of age, gender, occupation, and economic condition on mortality. In 1835 he attempted to apply probabilistic methods to the study of human characteristics, both physical and behavioural. He used them to give what he hoped was a complete description of the "average man." Although Quételet's work was generally highly respected, his attempt to apply it to the field of behavioural science was met with criticism. In the 1860s, the English scholar Francis Galton (1822–1911) attempted to apply statistics methods to the study of human heredity. His work was influential and helped define statistics as a mathematics discipline in its own right.

At the turn of the 20^{th} century, the corporate world began to recognize the relevance and usefulness of statistics, especially in issues of quality control, economics, insurance, and telecommunications. Many large companies began hiring statisticians.

While working for an English brewing company, the industrial scientist William Sealy Gosset (1876–1937) developed the Student’s t-test, allowing for the ability to derive reliable information from small samples. (Company policy forbade its employees to publish. Gosset did so in any case, writing under the pseudonym “Student”). The English mathematician Karl Pearson (1857–1936) developed the chi-squared test and is considered the founder of modern hypothesis testing.

Ronald Aylmer Fisher (1890–1962) is considered the most important statistician of the 20th century. His 1925 text “Statistical Methods for Research Workers” transformed statistics into a powerful scientific tool. He clarified many of the mathematical principles on which the discipline is based. Fisher also developed methods of multivariate analysis to properly analyze problems involving more than one variable.

In 1926, the pure and applied mathematician John von Neumann (1903–57) founded game theory — a mathematical framework for analyzing games of chance, such as poker, that involve strategy and choice on the parts of the players. Von Neumann recognized the applications of the theory to economics and social sciences. The work of Nobel Laureate John Forbes Nash, Jr., (1928–2015) took its applications to economics to a profound level.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 161. Identify the individuals based on the descriptions from the text in Activity 160.

Abraham de Moivre / Alexandr Lyapunov / Andrei Markov / Blaise Pascal / Carl Friedrich Gauss / Christiaan Huygens / Francis Galton / Girolamo Cardano / Jacob Bernoulli / John Forbes Nash, Jr. / John Graunt / John von Neumann / Joseph-Louis Lagrange / Karl Pearson / Lambert Adolphe Quételet / Pafnuty Chebyshev / Pierre de Fermat / Pierre-Simon Laplace / Ronald Aylmer Fisher / Siméon Denis Poisson / William Sealy Gosset

1. Analyzed birth and death records in 1662, producing the first life table.
2. A Belgian scholar who, in 1835, attempted to apply probabilistic methods to the study of human characteristics and made important observations about mortality.
3. Considered the most important statistician of the 20th century, transformed statistics into a powerful scientific tool.
4. A Dutch physicist-astronomer-mathematician who, in 1656, published "On Reasoning in Games of Chance," summarizing and extending ideas developed by Pascal and Fermat, introducing the notion of expected value.

5. An English mathematician who developed the chi-squared test and is considered the founder of modern hypothesis testing.
6. An English scholar who, in the 1860s, attempted to apply statistical methods to the study of human heredity.
7. A French mathematician who, in the latter half of the 19th century, contributed to the mathematical underpinnings of probability theory.
8. A French mathematician-astronomer-physicist who, in 1812, published "Analytical Theory of Probability," collecting and extending everything known on the subject at that time.
9. A French mathematician-astronomer-physicist who, in the latter half of the 19th century, contributed to the theoretical foundations of probability theory.
10. A German genius who, in the latter half of the 19th century, contributed to the theoretical foundations of probability theory.
11. An industrial scientist who, while working for an English brewing company, developed the Student’s t-test.
12. An Italian mathematician and physician who, in 1663, posthumously had his work "Book on Games of Chance" published, recognized the law of large numbers.
13. A mathematician who collaborated with Blaise Pascal to solve Chevalier de Méré's "problem of points," contributing to the development of probability theory.
14. A mathematician who received a letter from Chevalier de Méré in 1654, leading to the birth of probability theory in collaboration with Pierre de Fermat.
15. A mathematician who recognized the repeated appearance of the normal distribution in scientific studies in 1733.
16. A Nobel Laureate whose work took game theory applications to economics to a profound level.
17. A pure and applied mathematician who founded game theory in 1926, recognizing applications to economics and social sciences.
18. A Russian mathematician who, in the late 19th century, further developed the mathematical underpinnings of probability theory.
19. A Swiss mathematician who recognized the wide-ranging applicability of probability in fields outside of gambling; his posthumously published book in 1713 demonstrated the use of probability theory in medicine and meteorology.

Activity 162. Rearrange the events in chronological order according to the text of Activity 160. Provide dates where possible.

a. Abraham de Moivre recognizes the normal distribution in scientific studies and writes a mathematical equation for it.

b. Athens conducts a population census.
c. Chevalier de Méré writes to Blaise Pascal seeking advice on divvying up stakes from interrupted games.
d. Christiaan Huygens publishes "On Reasoning in Games of Chance," introducing the notion of expected value.
e. Fisher's text clarifies mathematical principles in statistics.
f. Francis Galton attempts to apply statistical methods to the study of human heredity.
g. Girolamo Cardano's "Book on Games of Chance" is published, recognizing the law of large numbers.
h. Greeks take a census for tax purposes.
i. Jacob Bernoulli's posthumous book "The Art of Conjecture" demonstrates the use of probability theory in medicine and meteorology.
j. John Graunt analyzes birth and death records, producing the first life table.
k. John von Neumann founds game theory.
l. Pierre-Simon Laplace publishes "Analytical Theory of Probability."
m. The corporate world recognizes the relevance and usefulness of statistics, leading to the hiring of statisticians.
n. The United States takes its first decennial census.

Activity 163. Determine whether the statements are true or false by quoting from the text in Activity 160.

1. A substantial number of statistical records were maintained during the Middle Ages.
2. Abraham de Moivre, in 1733, identified the recurrent presence of the normal distribution in scientific studies and formulated a mathematical equation for it.
3. In the early 20th century, businesses started acknowledging the importance and utility of statistics, particularly in quality control, economics, insurance, and telecommunications.
4. Jacob Bernoulli from the renowned Bernoulli family acknowledged the broad applicability of probability beyond gambling.
5. John von Neumann established game theory in 1916.
6. Karl Pearson developed the Student’s t-test.
7. Pascal and Fermat's discussions contributed to the birth of mathematical statistics.
8. Probability theory's fundamental principle asserts that if situations have equally likely possible outcomes, the probability of a specific outcome is 1 divided by the total number of outcomes.
9. The development of probability theory was significantly driven by inquiries in betting and gaming.
10. The life table was created without the intention of making general observations and predictions about life expectancy for specific population classes.

Activity 164. In groups, discuss the points. Refer to the text in Activity 160.

1. How did questions in betting and gaming provide the initial motivation for the development of probability theory, as seen in the interactions between Chevalier de Méré and Blaise Pascal?
2. How did Pascal and Fermat's collaboration, initiated by the concerns raised by de Méré, lead to the birth of probability theory, and what were the key elements of their solutions to the "problem of points"?
3. How did Christiaan Huygens contribute to the development of probability theory, and what role did the notion of expected value play in summarizing and extending the ideas of Pascal and Fermat?
4. What is the key principle behind probability theory, as articulated by Girolamo Cardano, and how did it lay the groundwork for the later work of Pascal and Fermat?
5. How did Jacob Bernoulli demonstrate the wide-ranging applicability of probability in fields beyond gambling, particularly in medicine and meteorology, through his book "The Art of Conjecture"?
6. Explore the relationship between probability and statistics, considering examples such as the known outcomes from a pair of dice and the inference about teeth-brushing habits of a population based on a small sample.
7. Delve into how Abraham de Moivre recognized the significance of the normal distribution, initially observed from errors in astronomical observations, and its mathematical representation.
8. Examine the contributions of Lagrange, Laplace, Gauss, and Poisson in the 19th century, including their mathematical proof of the law of large numbers and Laplace's pivotal work in the 1812 text "Analytical Theory of Probability."
9. Trace the historical development of statistics from ancient civilizations to John Graunt's analysis of birth and death records in 1662, highlighting key milestones in the collection and analysis of numerical information.
10. Explore the recognition of statistics in the corporate world during the 20th century, the contributions of statisticians like Gosset, Pearson, and Fisher, and the transformative impact of Fisher's 1925 text on statistical methods for research workers.

Activity 165. In writing, prove that probability theory and mathematical statistics represent two sides of the same coin. Refer to the text in Activity 160.

“Eureka!” (Archimedes of Syracuse)