“Eureka!” (Archimedes of Syracuse)

Module 5. History of Mathematics

Unit 21. Babylonian Mathematics

Activity 166. Do the quiz on Babylonian mathematics. In pairs, compare your answers.

1. **Where did the Babylonians of 2000 B.C.E. live?**
   a. Greece
   b. Mesopotamia
   c. Egypt
   d. Rome
2. **What material did the Babylonians use for record-keeping?**
   a. papyrus
   b. stone
   c. clay tablets
   d. animal skins
3. **How did Babylonians represent numbers greater than 59?**
   a. additive system
   b. binary system
   c. base-60 place-value system
   d. Roman numerals
4. **Why did Babylonians choose a sexagesimal system?**
   a. It is highly divisible.
   b. It reflects lunar cycles.
   c. It is based on prime numbers.
   d. It aligns with the number of fingers and toes.
5. **What ambiguity arose in the Babylonian numeral system?**
   a. lack of symbols
   b. symbol shapes
   c. no zero symbol
   d. no fraction representation
6. **How did Babylonians solve quadratic equations?**
   a. linear equations

   b. reciprocal tables
   c. Pythagorean theorem
   d. famous quadratic formula

7. **What method did Babylonians use for cubic equations?**
   a. division
   b. multiplication
   c. setting variables
   d. Pythagorean theorem
8. **What did Babylonians use to compute the diagonal length of a square?**
   a. arithmetic mean
   b. Pythagoras’s theorem
   c. trigonometry
   d. reciprocal tables
9. **What evidence suggests Babylonians enjoyed mathematics for its own sake?**
   a. practical applications
   b. Pythagorean triples
   c. tables of powers
   d. linear equations
10. **How did Babylonians approximate the square root of two?**
   a. trial and error
   b. Heron’s method
   c. linear interpolation
   d. exponential growth

Activity 167. Read the article. Review your answers to the quiz in Activity 166.

The Babylonians of 2000 B.C.E. lived in Mesopotamia, the fertile plain between the Euphrates and Tigris Rivers in what is now Iraq. We are fortunate that the peoples of this region kept extensive records of their society — and their mathematics — on hardy sun-baked clay tablets. A large number of these tablets survive today. The Babylonians used a simple stylus to make marks in the clay and developed a form of writing based on cuneiform (wedge-shaped) symbols.

The mathematical activity of the Babylonians seems to have been motivated, at first, by the practical everyday needs of running their society. Many problems described in early tablets are concerned with calculating the number of workers needed for building irrigation canals and the total expense of wages, for instance. But many problems described in later texts have no apparent practical application and clearly indicate an interest in pursuing mathematics for its own sake.

The Babylonians used only two symbols to represent numbers: the symbol to represent a unit and the symbol to represent a group of ten. A simple additive system was used to represent the numbers 1 through 59. A base-60 place-value system was then used to represent numbers greater than 59. Spaces were inserted between clusters of symbols.

Historians are not clear as to why the Babylonians chose to work with a sexagesimal system. A popular theory suggests that this number system is based on the observation that there are 365 days in the year. When rounded to the more convenient (highly divisible) value of 360, we have a multiple of 60. Vestiges of this number system remain with us today. For example, we use the number 360 for the number of degrees in a circle, and we count 60 seconds in a minute and 60 minutes per hour.

There were two points of possible confusion with the Babylonian numeral system. With no symbol for zero, it is not clear whether the numeral represents 61 (as one unit of 60 plus a single unit), 3601 (as one unit of 602 plus a single unit), or even 216,060, for instance. Also, the Babylonians were comfortable with fractions and used negative powers of 60 to represent them (just as we use negative powers of 10 to write fractions in decimal notation). But with no notation for the equivalent of a decimal point, the symbol could also be interpreted to mean 1 + (1/60), or (1/60) + (1/602), or even 60 + (1/604), for instance. As the Babylonians never developed a method for resolving such ambiguity, we assume then that it was never considered a problem for scholars of the time. (Historians suggest that the context of the text always made the interpretation of the numeral apparent.)

The Babylonians compiled extensive tables of powers of numbers and their reciprocals, which they used in ingenious ways to perform arithmetic computations. (For instance, a tablet dated from 2000 B.C.E. lists all the squares of the numbers from one to 59, and all the cubes of the numbers from one to 32.) To compute the product of two numbers “a” and “b”, Babylonian scholars first computed their sum and their difference, read the squares of those numbers from a table, and divided their difference by four. (In modern notation, this corresponds to the computation: $ab = (1/4)\ [(a + b)^2 - (a - b)^2]$.) To divide a number “a” by “b”, scholars computed the product of “a” and the reciprocal 1/b (recorded in a table): $ab = a \times (1/b)$. The same table of reciprocals also provided the means to solve linear equations: $bx = a$. (Multiply “a” by the reciprocal of “b”.)

Problems in geometry and the computation of area often lead to the need to solve quadratic equations. For instance, a problem from one tablet asks for the width of a rectangle whose area is 60 and whose length is seven units longer than the width. In modern notation, this amounts to solving the equation $x(x + 7) = x^2 + 7x = 60$. The scribe who wrote the tablet then proffers a solution that is equivalent to the famous quadratic formula: $x = \sqrt{(7/2)^2 + 60} - (7/2) = 5$. (Square roots were computed by examining a table of squares.)

Problems about volume lead to cubic equations, and the Babylonians were adept at solving special equations of the form: $ax^3 + bx^2 = c$. (They solved these by setting $n = (ax)/b$, from which the equation can be rewritten as $n^3 + n^2 = ca^2/b^3$. By examining a table of values for $n^3 + n^2$, the solution can be deduced.)

It is clear that Babylonian scholars knew of Pythagoras's theorem, although they wrote no general proof of the result. If the width of a rectangle is four units and the length of its diagonal is five units, what is its breadth? Four times four is 16, and five times five is 25. Subtract 16 from 25 and there remains nine. What times what equals nine? Three times three is nine. The breadth is three.

The Babylonians used Pythagoras's theorem to compute the diagonal length of a square, and they found an approximation to the square root of two accurate to five decimal places. (It is believed that they used a method analogous to Heron's method to do this.) Babylonian scholars were also interested in approximating the areas and volumes of various common shapes by using techniques that often invoked Pythagoras's theorem.

Most remarkable is a tablet that lists 15 large Pythagorean triples. As there is no apparent practical need to list these triples, this strongly suggests that the Babylonians did indeed enjoy mathematics for its own sake.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 168. Determine whether the statements are true or false by quoting from the text in Activity 167.

1. Babylonian scholars compiled extensive tables of powers and reciprocals, showcasing sophisticated arithmetic methods, encompassing solving linear, quadratic, and cubic equations.
2. Babylonians developed a systematic approach to resolve ambiguity in numeral interpretation, ensuring precision in their mathematical writings.
3. Babylonians opted for the sexagesimal system due to its mathematical convenience, with historians providing a clear understanding of this choice.
4. Employing intricate cuneiform symbols, the Babylonians had a diverse set of symbols for each number in their numerical system.
5. Initially driven by abstract pursuits, Babylonian mathematical challenges later incorporated practical applications in their texts.
6. The Babylonian numeral system lacked a zero symbol, leading to ambiguity, and they utilized negative powers of 60 for fractions with no decimal point.
7. The Babylonians, residing in Mesopotamia circa 2000 B.C.E., meticulously recorded their society and mathematical endeavours on robust clay tablets.
8. Utilizing a base-10 place-value system, Babylonians employed symbols for all numbers between 1 and 59, simplifying their numerical representation.

Activity 169. In groups, discuss the points. Refer to the text in Activity 167.

1. Discuss the shift in the Babylonians' mathematical focus from practical everyday needs, such as building irrigation canals, to abstract pursuits for the sake of mathematics itself. What might have prompted this transition?
2. Explore the simplicity of the Babylonian numeral system with only two symbols. How might such a system, based on units and groups of ten, compare to or differ from modern numerical representations?
3. Delve into the theories surrounding the Babylonians' choice of the sexagesimal system. How did the observation of 365 days in a year influence their numerical system, and what vestiges of this system are still present in contemporary measurements?
4. Discuss the challenges posed by the Babylonian numeral system, particularly the absence of zero and a decimal point. How might this ambiguity have impacted their mathematical calculations, and why do historians believe it wasn't a concern for Babylonian scholars?
5. Examine the Babylonians' use of tables listing powers of numbers and their reciprocals. How did this approach aid them in performing arithmetic computations, and how does it compare to modern computational methods?
6. Explore the Babylonians' methods for solving quadratic and cubic equations. How do their approaches, such as using squares and cubes tables, compare to contemporary methods for solving similar equations?
7. Discuss the Babylonians' application of Pythagoras’s theorem in practical problem-solving, such as finding the breadth of a rectangle. How did they employ this theorem in areas beyond geometry, like computing square roots?
8. Explore how the Babylonians approximated the square root of two and applied Pythagoras’s theorem to various shapes. How did these approximations demonstrate their interest in real-world measurements and calculations?
9. Investigate the significance of the tablet listing 15 large Pythagorean triples. Why might the Babylonians have documented these triples, and what does it reveal about their enjoying mathematics for its own sake?
10. Discuss the lasting impact of Babylonian mathematics on contemporary mathematical practices. In what ways have their methods influenced modern mathematical systems and approaches?

Activity 170. Write an overview of the text in Activity 167 using Appendix II.

Unit 22. Egyptian Mathematics

Activity 171. Do the quiz on Egyptian mathematics. In pairs, compare your answers.

1. **Where does our knowledge of ancient Egyptian mathematics primarily come from?**
   a. Rosetta Stone
   b. Rhind papyrus
   c. pyramid inscriptions
   d. Library of Alexandria
2. **What was the Egyptian approach to denoting numerals like?**
   a. place-value system
   b. Roman numeral system
   c. hieroglyphic symbols
   d. decimal system
3. **How did Egyptians perform arithmetic calculations without a place-value system?**
   a. mental calculations
   b. using an abacus-like calculating board
   c. with pen and paper
   d. utilizing complex algorithms
4. **What method did the ancient Egyptians use for multiplication?**
   a. long division
   b. successive doubling
   c. Egyptian addition
   d. false position
5. **How did Egyptians express fractions in the Ahmes papyrus?**
   a. with numerators and denominators
   b. using a dot over the number
   c. through symbols only
   d. writing fractions in words
6. **What term is used today for fractions with unit numerators?**
   a. Pythagorean fractions
   b. Hieratic fractions
   c. Greek fractions
   d. Egyptian fractions

7. **What method did the Egyptians employ to solve linear equations, as described in the Ahmes papyrus?**
    a. trial and error
    b. calculus
    c. false position
    d. matrix algebra
8. **What kind of problems make up the majority of the Ahmes papyrus?**
    a. algebraic problems
    b. number theoretic problems
    c. practical problems related to area, volume, and more
    d. geometric proofs
9. **Which problem in the Ahmes papyrus reflects a delight in mathematical thinking for its own sake?**
    a. problem 1: numerical notation
    b. problem 24: false position
    c. problem 50: multiplication techniques
    d. problem 79: houses, cats, mice, and grain

Activity 172. Read the article. Review your answers to the quiz in Activity 171.

Our knowledge of ancient Egyptian mathematics from around 2000 B.C.E. comes chiefly from the Rhind papyrus (Ahmes papyrus). There we learn, for example, that the Egyptians followed a very natural system for denoting numerals: 1 was a vertical stroke |, 2 was two of them ||, 3 was |||, and 4 was ||||, and separate symbols were used for 5, 6, 7, 8, and 9, and for 10, 20, ..., 100, 200, ..., 1000, and so on. All other numbers were represented as groups of these symbols, usually arranged in order from largest to smallest. Like the Roman numeral system, the Egyptian system did not use a place-value system (the symbol for 5, for example, denoted "5" no matter where it appeared in the number). It is very difficult to do pencil-and-paper calculations without place-value notation, but the Egyptians always used a calculating board, much like an abacus, to perform arithmetic calculations, and needed only to record the results. They were therefore not hindered by their cumbersome numerical system. The ancient Egyptians were adept at multiplication, using a method of successive doubling to calculate products. This method is today called Egyptian multiplication.

Division problems lead to fractions. It did not occur to the ancient Egyptians to express fractions with numerators and denominators. In the Rhind papyrus, the mathematician Ahmes simply placed a dot over a number to indicate its reciprocal, except in the case of the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{1}{4}$, each of which had its own symbol. Thus, the Egyptians only dealt with fractions of the form $\frac{1}{n}$ (with the exception of two-thirds). Fractions with unit numerators are known today as Egyptian fractions. All other fractional quantities were expressed as sums of distinct Egyptian fractions. The Egyptian's ability to compute such expressions is impressive. The Rhind papyrus provides reference lists of such expressions, and the first 23 problems in the document are exercises in working with such fractional representations.

The ancient Egyptians were adept at solving linear equations. They used a method called false position to attain solutions. This involves guessing an answer, observing the outcome from the guess, and adjusting the guess accordingly. As an example, problem 24 of the Rhind papyrus asks:

Find the quantity so that when 1/7 of itself is added to it, the total is 19.

To demonstrate the solution, the author suggests a guess of 7. That plus its one-seventh is 8, by far too small, but multiplying the outcome by 19/8 produces the answer of 19 that we need. Thus, 7 × (19/8) must be the quantity we desire.

The majority of problems in the Rhind papyrus are practical in nature, dealing with issues of area (of rectangles, trapezoids, triangles, circles), volume (of cylinders, for example), slopes and altitudes of pyramids (which were built 1,000 years before the text was written), and number theoretic problems about sharing goods under certain constraints. Some problems, however, indicate a delight in mathematical thinking for its own sake. For example, problem 79 asks:

If there are seven houses, each house with seven cats, seven mice for each cat, seven ears of grain for each mouse, and each ear of grain would produce seven measures of grain if planted, how many items are there altogether?

This problem appears in Fibonacci's "The Book of the Abacus", written 600 years before the Rhind papyrus was discovered. A version of this problem also appears as a familiar nursery-rhyme and riddle, "As I Was Going to St. Ives."

(by James Tanton, from Encyclopedia of Mathematics)

Table 29

Rhind Papyrus	**Nursery Rhyme**
If there are seven houses, each house with seven cats, seven mice for each cat, seven ears of grain for each mouse, and each ear of grain would produce seven measures of grain if planted, how many items are there altogether?	As I was going to St. Ives, I met a man with seven wives, Each wife had seven sacks, Each sack had seven cats, Each cat had seven kits: Kits, cats, sacks, and wives, How many were there going to St. Ives?

Activity 173. Determine whether the statements are true or false by quoting from the text in Activity 172.

1. Ancient Egyptians excelled in multiplication, employing a technique of successive doubling for calculations.
2. Ancient Egyptians were aware of expressing fractions with numerators and denominators.
3. Ancient Egyptians were skilled in solving quadratic equations through the method of false position.
4. Egyptians did not always rely on a calculating board or abacus for arithmetic calculations.
5. Egyptians used an intuitive system to represent numerals.
6. Most problems in the Rhind papyrus are practical, involving concepts like area, volume, and the characteristics of pyramids.
7. Not all fractional quantities were represented as sums of distinct Egyptian fractions.
8. The primary source of information on ancient Egyptian mathematics around 2000 B.C.E. is the Rhind papyrus, also known as the Ahmes papyrus.
9. The term "Egyptian fractions" refers to fractions with unit denominators.

Activity 174. In groups, discuss the points. Refer to the text in Activity 172.

1. Discuss the unique system of numerical notation used by the ancient Egyptians, highlighting the symbolic representation of numbers and the absence of a place-value system.
2. Explore the significance of calculating boards, similar to abacuses, in Egyptian mathematics. Discuss how these tools allowed efficient arithmetic calculations despite the limitations of their numerical system.
3. Examine the method of successive doubling employed by the Egyptians for multiplication. Discuss the implications of this technique and its relevance in contemporary mathematics.
4. Delve into the Egyptian approach to fractions, emphasizing the use of symbols and the specific treatment of fractions like 1/2, 1/3, 2/3, and 1/4. Discuss the concept of Egyptian fractions and their representation.
5. Explore the method of false position used by the ancient Egyptians to solve linear equations. Discuss the problem-solving approach involving guesswork and adjustments, using a specific example from the Rhind papyrus.
6. Analyze the majority of problems in the Rhind papyrus, which have practical applications in geometry, volume calculations, and number theory. Discuss the real-world scenarios covered in these mathematical problems.

7. Highlight the impressive computational skills of the ancient Egyptians, as evidenced by their ability to compute complex expressions and their reliance on reference lists for problem-solving.
8. Explore instances in the Rhind papyrus where mathematical problems indicate a delight in mathematical thinking for its own sake. Discuss the significance of such problems in understanding the mathematical mindset of the ancient Egyptians.
9. Compare the mathematical problems in the Rhind papyrus with similar problems found in other mathematical texts, such as Fibonacci's "The Book of the Abacus." Discuss the cross-cultural aspects of mathematical problem-solving.
10. Discuss the enduring legacy of Egyptian mathematics, considering its influence on subsequent mathematical developments and its relevance in modern mathematical discourse.

Activity 175. Write an overview of the text in Activity 172 using Appendix II.

Unit 23. Greek Mathematics

Figure 9. "The School of Athens" (by Raphael)

Activity 176. Look at the image in Figure 9. In pairs, discuss the questions.

1. Who was Raphael?
2. What scholars and scientists are depicted in the painting? What are they famous for?
3. How does the fresco reflect the values of the Renaissance?

Activity 177. Study Table 30. Illustrate how different letters of the Greek alphabet are used in mathematics.

Table 30. Greek Alphabet

№	Letter	Name		№	Letter	Name	
1	Α α	alpha	/ˈælfə/	13	Ν ν	nu	/njuː/
2	Β β	beta	/ˈbiːtə/ /ˈbeɪtə/ (Am)	14	Ξ ξ	xi	/zaɪ/ /ksaɪ/
3	Γ γ	gamma	/ˈgæmə/	15	Ο ο	omicron	/ˈɒmɪkrɒn/ /oʊˈmaɪkrɒn/ (Br)
4	Δ δ	delta	/ˈdɛltə/	16	Π π	pi	/paɪ/
5	Ε ε	epsilon	/ˈɛpsɪlɒn/ /ɛpˈsaɪlən/ (Br)	17	Ρ ρ	rho	/roʊ/
6	Ζ ζ	zeta	/ˈziːtə/ /ˈzeɪtə/ (Am)	18	Σ σ/ς C c	sigma	/ˈsɪgmə/
7	Η η	eta	/ˈiːtə/ /ˈeɪtə/ (Am)	19	Τ τ	tau	/taʊ/ /tɔː/
8	Θ θ	theta	/ˈθiːtə/ /ˈθeɪtə/ (Am)	20	Υ υ	upsilon	/juːpˈsaɪlən/ /ˈʊpsɪlɒn/ /ʌpˈsaɪlən/ (Br) /ˈʌpsɪlɒn/ (Am)
9	Ι ι	iota	/aɪˈoʊtə/	21	Φ ϕ	phi	/faɪ/
10	Κ κ	kappa	/ˈkæpə/	22	Χ χ	chi	/kaɪ/
11	Λ λ	lambda	/ˈlæmdə/	23	Ψ ψ	psi	/saɪ/ /psaɪ/
12	Μ μ	mu	/mjuː/ /muː/ (Am)	24	Ω ω	omega	/ˈoʊmɪgə/ (Br) /oʊˈmeɪgə/ (Am)

Activity 178. Do the quiz on Greek mathematics. In pairs, compare your answers.

1. **Who is considered the first Greek mathematician of note?**
   a. Euclid
   b. Thales of Miletus
   c. Archimedes

   d. Pythagoras
2. **What did Euclid's work "The Elements" primarily focus on?**
   a. conic sections
   b. trigonometry
   c. deductive reasoning and logical proofs
   d. squaring the circle
3. **In Greek mathematics, how was a "number" conceptualized?**
   a. a line segment
   b. a point
   c. a circle
   d. an angle
4. **What did Plato use as an example of something that cannot be discovered by the senses but can be discovered by logical reasoning?**
   a. calculus
   b. geometry
   c. mathematics
   d. trigonometry
5. **Who is considered the most influential mathematics scholar of all time in Greek mathematics?**
   a. Archimedes
   b. Plato
   c. Euclid
   d. Pythagoras
6. **What did Archimedes of Syracuse solve, and what method did he use?**
   a. squaring the circle, using a straightedge and compass
   b. trisecting an angle, using an Archimedean spiral
   c. squaring the parabola, using a straightedge and compass
   d. duplicating the cube, using a conic section
7. **Who continued work on conic sections and properly defined an ellipse, a hyperbola, and a parabola?**
   a. Hipparchus
   b. Diophantus
   c. Ptolemy
   d. Apollonius of Perga
8. **What did Claudius Ptolemy attempt to prove in the 2nd century C.E.?**
   a. Euclid’s parallel postulate
   b. Pythagoras’s theorem
   c. Archimedes's principle
   d. Hipparchus's table

Activity 179. Read the article. Review your answers to the quiz in Activity 178.

The ancient Greeks of ca. 600 B.C.E. to ca. 480 C.E. set the current standards of logical rigour in mathematics. Although many ancient cultures practiced and developed mathematics, it was the Greeks who developed the explicit art of “proof” and explored the power of pure deductive reasoning to its fullest.

We should mention that when speaking of “Greek mathematics,” historians include any mathematician who wrote in the Greek language and followed the Greek tradition of mathematical thought. Greek was the common language of the Mediterranean world during ancient times, and many intellectuals from different parts of that region are today considered Greek scholars. For instance, the great Archimedes was from Syracuse, now a part of Italy, and Euclid (ca. 300–260 B.C.E.) is believed to have lived in Alexandria, Egypt.

There are very few original records of Greek work. Initially, knowledge was transmitted only orally from teacher to student. Around 450 B.C.E. the Greeks adopted the ancient Egyptian practice of writing on papyrus scrolls. Unfortunately, papyrus — a grasslike plant grown in the Nile Delta region — decays rapidly away from the exceptionally dry climate of Egypt. The Greeks combated this problem by repeatedly making copies of their works but, because of the effort involved, copied only those pieces they deemed of utmost importance. The first mathematical work preserved and honoured this way was Euclid’s masterpiece “The Elements” of ca. 300 B.C.E. Historians have had to rely on commentary made by later scholars to deduce what was accomplished mathematically before the time of Euclid.

Greek scholars approached all of mathematics through the study of geometry. Even their work on the properties of whole numbers, ratios, and proportions, as well as mechanics and astronomy was done in a geometric style. A “number,” for instance, was literally a line segment, and a “ratio” was understood in terms of commensurable segments. It is interesting to note that Greek scholars took careful steps to avoid speaking directly of the infinite. (The 5th-century B.C.E. paradoxes on the nature of motion and the infinitely small developed by Zeno of Elea deeply affected Greek thinking.) For instance, Euclid stated that any line segment could be extended to any arbitrary length, but never spoke of lines that were infinitely long. In Euclid’s proof of the infinitude of primes, Euclid stated that from any finite list of prime numbers one can always construct one more, but never spoke of the set of primes as infinite.

Many historians regard Thales of Miletus (ca. 625–547 B.C.E.) as the first Greek mathematician of note. Commentaries suggest that Thales identified, and proved, seven key geometric propositions, including that the base angles of an isosceles triangle are always equal and that the inscribed angle from the diameter of a circle is always a right angle, for instance. The great scholar and mystic Pythagoras lived a century later, and he and his followers are credited with the discovery of the famous result about right triangles (today called Pythagoras’s theorem) and the discovery of irrational numbers. A great deal of mystery surrounds the life and legend of Pythagoras. He founded a semireligious sect called the Pythagorean Brotherhood (women were equal members) based on certain mystic significances ascribed to whole numbers and their ratios.

The great philosopher Plato (428–348 B.C.E.) wrote a great deal about mathematics in his famous dialogues, demonstrating a deep personal respect for the subject. The five regular polyhedra — the Platonic solids — are named in his honour. In his philosophical treatises, Plato used the example of mathematics as something that cannot be discovered by the senses but can nonetheless be discovered by the power of logical reasoning. He also believed mathematics to be an essential part of a cultured person's education. Philosopher Aristotle (384–322 B.C.E.) adopted the same view and used mathematics as examples in his development of formal logic and his analysis of arguments.

Today, the Greek scholar Euclid is considered to be the most influential mathematics scholar of all time. In his famous work "The Elements", Euclid collated all mathematical knowledge known at his time into a single tome. Although an impressive feat, it was the organization of the text that had the greatest impact. Beginning with a small collection of "self-evident truths," Euclid showed that all mathematical knowledge of his time could be deduced by pure logical reasoning alone. This work demonstrated the power of the mind and set the model for all mathematical research in the future. Mathematicians today still work to the standards of rigour as set by Euclid. Next to the Bible, Euclid's "The Elements" is the most widely published book of all time.

After producing "The Elements", Euclid continued work on the conic sections, on optics, and on general problems in geometry. He continued interest in constructible numbers and no doubt contemplated the classic Greek problem of squaring the circle. (In "The Elements" Euclid had demonstrated general procedures for squaring arbitrary polygonal figures.) This challenge, as well as the problems of trisecting an angle and duplicating the cube, spurred a great deal of significant further research in mathematics for centuries to come.

Archimedes of Syracuse (ca. 287–212 B.C.E.) solved the problem of squaring the parabola, as well as made significant advances in computing the areas and volumes of curved figures and solids. (He also "solved" the problem of squaring the circle by making use of his Archimedean spiral. Unfortunately, his method went beyond the use of a straightedge and compass alone, and so is not a permissible solution to the original problem.)

Apollonius of Perga (ca. 262–190 B.C.E.) continued work on conic sections and is credited for properly defining an ellipse, a hyperbola, and a parabola. Around the same time, Greek astronomer Hipparchus wrote a table of "chord values" (the equivalent to a modern table of sine values), which he used to solve astronomical problems. This represented the beginning development of trigonometry in Greek mathematics, but also marked an end of fervent mathematical development in the Greek tradition. For the five centuries that followed, new developments were limited to straightforward advances in astronomy, trigonometry, and algebra, with just a few notable exceptions.

In the 2nd century C.E., The Greek astronomer Claudius Ptolemy corrected and extended Hipparchus's table and clarified the mathematics that is used to produce such a table. He is also known as one of the first scholars to make a serious attempt at proving Euclid's parallel postulate. In the 3rd century, Diophantus of Alexandria produced his famous text "Arithmetic", from which the study of Diophantine equations was born. In the mid-4th century, the enthusiastic Pappus of Alexandria attempted to revive interest in ardent mathematical research of the Greek style. He produced his treatise "Synagoge" (Collection) to act as a commentary and guide to all the geometric works of his time and included in it a significant number of original results, extensions of ideas, and innovative shifts of perspective. Unfortunately, he did not succeed in his general goal. After Pappus, of note is Hypatia of Alexandria (370–415), the first woman to be named in the history of mathematics, credited for writing insightful commentaries on the works of Apollonius and Diophantus, and Proclus (ca. 410–485), who is noted for his detailed commentary on the work of Euclid and his own attempt to prove the parallel postulate.

The beginning of the 5th century marks a clear end to the tradition of Greek mathematics.

(by James Tanton, from Encyclopedia of Mathematics)

Table 31. Regular Polyhedra

Platonic Solid	tetrahedron	cube	octahedron	dodecahedron	icosahedron
Faces	4	6	8	12	20

Activity 180. Identify the individuals based on the descriptions from the text in Activity 179.

Archimedes / Aristotle / Claudius Ptolemy / Diophantus of Alexandria / Euclid / Hypatia of Alexandria / Pappus of Alexandria / Plato / Proclus / Pythagoras / Thales of Miletus

1. An Enthusiastic mathematician of the mid-4th century who attempted to revive interest in mathematical research, produced the treatise "Synagoge" (Collection).

2. First Greek mathematician of note, identified and proved seven key geometric propositions.
3. First woman named in the history of mathematics, credited for insightful commentaries on the works of Apollonius and Diophantus.
4. Great mathematician from Syracuse, now part of Italy.
5. Great philosopher who wrote extensively about mathematics in his famous dialogues, honoured with the naming of the five regular polyhedra.
6. Greek astronomer of the 2nd century C.E., corrected and extended Hipparchus's table, made a serious attempt at proving Euclid's parallel postulate.
7. Greek scholar believed to have lived in Alexandria, Egypt, considered the most influential mathematics scholar of all time.
8. Mathematician noted for a detailed commentary on the work of Euclid and his attempt to prove the parallel postulate.
9. Mathematician of the 3rd century, produced the famous text "Arithmetic," from which the study of Diophantine equations was born.
10. Mystic and scholar credited with the discovery of Pythagoras's theorem and irrational numbers, founded the Pythagorean Brotherhood.
11. Philosopher who used mathematics as examples in the development of formal logic and analysis of arguments.

Activity 181. Rearrange the events in chronological order according to the text of Activity 179. Provide dates where possible.

a. Apollonius continues work on conic sections, properly defining an ellipse, hyperbola, and parabola.
b. Archimedes solves the problem of squaring the parabola and makes significant advances in computing the areas and volumes of curved figures and solids.
c. Aristotle adopts Plato's view on mathematics and incorporates it into his development of formal logic and analysis of arguments.
d. Diophantus produces his famous text "Arithmetic," giving birth to the study of Diophantine equations.
e. Euclid compiles "The Elements," organizing all mathematical knowledge of his time into a single tome.
f. Greek mathematics experiences a decline, marking the end of the tradition that set the standards for logical rigour and deductive reasoning.
g. Hypatia becomes the first woman named in the history of mathematics, credited for insightful commentaries on the works of Apollonius and Diophantus.

h. Pappus attempts to revive interest in Greek-style mathematical research with his treatise "Synagoge" serving as a commentary and guide to geometric works, including original results and innovative perspectives.
i. Plato writes extensively about mathematics in his dialogues, introduces the concept of mathematics as something discoverable through logical reasoning, not the senses.
j. Proclus provides a detailed commentary on Euclid's work and attempts to prove the parallel postulate.
k. Ptolemy corrects and extends Hipparchus's table of "chord values.", makes a serious attempt to prove Euclid's parallel postulate, contributing to the understanding of geometry.
l. Pythagoras and his followers discover Pythagoras's theorem and irrational numbers.
m. Thales identifies and proves key geometric propositions, lays the foundation for Greek mathematical exploration.

Activity 182. Determine whether the statements are true or false by quoting from the text in Activity 179.

1. Archimedes of Syracuse solved the problem of squaring the circle using only a protractor in adherence to classical Greek geometric methods.
2. Aristotle, the philosopher, expressed a profound respect for mathematics in his dialogues and considered it a vital component of a cultured education.
3. Claudius Ptolemy and his followers are credited with the famous result concerning right triangles.
4. Euclid, in his proof of the infinitude of primes, explicitly mentioned the set of primes as infinite.
5. Euclid's "The Elements" stands as the most influential mathematical work, organizing knowledge and establishing enduring standards of rigour.
6. Historians classify as Greek mathematicians those who wrote in Greek and followed the Greek mathematical tradition, regardless of their geographic origins.
7. Thales of Miletus is often recognized as the earliest noteworthy Greek mathematician, acknowledged for proving essential geometric propositions.
8. The Greeks adopted the practice of writing on papyrus scrolls around 450 B.C.E., a departure from the ancient Egyptian method.
9. The Greeks significantly shaped the logical rigour in mathematics from around 600 B.C.E. to 480 C.E.

Activity 183. In groups, discuss the points. Refer to the text in Activity 179.

1. Explore the fundamental contributions of ancient Greek mathematicians, such as Euclid, to geometry. Discuss the development of Euclidean geometry and its lasting impact on mathematical thinking.
2. Delve into the Pythagorean theorem and its origins within the Pythagorean school. Discuss its applications, proofs, and significance in both geometry and mathematics as a whole.
3. Examine the work of Archimedes, particularly in the field of mathematical physics. Discuss Archimedes' principle, the law of the lever, and his methods for calculating areas and volumes.
4. Discuss the contributions of ancient Greek mathematicians, such as Zeno and Eudoxus, to the understanding of infinite series and calculus precursors. Explore the challenges they faced in dealing with the concept of infinity.
5. Explore the philosophical aspects of mathematics in ancient Greece, particularly the views of Plato and Aristotle. Discuss how their philosophical ideas influenced the development of mathematical thought.
6. Discuss the advancements in mathematics during the Hellenistic period, including contributions from mathematicians like Apollonius of Perga. Explore the developments in conic sections and other geometric theories.
7. Examine the mathematical notations used by ancient Greek mathematicians. Discuss the symbols, methods of representation, and how these notations evolved over time.
8. Explore the role of mathematics in ancient Greek education. Discuss how mathematical concepts were taught, the significance of mathematics in the curriculum, and its impact on intellectual development.
9. Investigate the contributions of ancient Greek mathematicians to number theory: the study of prime numbers, perfect numbers, and the sieve of Eratosthenes.

Activity 184. Write an overview of the text in Activity 179 using Appendix II.

Unit 24. Indian Mathematics

Activity 185. Do the quiz on Indian mathematics. In pairs, compare your answers.

1. **What was the significant Indian invention that profoundly influenced Western mathematics?**
   a. Egyptian numeral system
   b. number zero
   c. place-value decimal system
   d. Roman numeral system
2. **How many symbols were used in the Indian place-value system to represent numbers?**
   a. 5
   b. 7
   c. 10
   d. 12
3. **What was the basic unit of length used by the Indus civilization in their decimal system?**
   a. Indus meter
   b. Roman inch
   c. Indus inch
   d. Egyptian cubit
4. **During which period was mathematics in India driven by the needs of Jainism, focusing on accurate astronomical observations?**
   a. Vedic period
   b. Maurya period
   c. Gupta period
   d. Jainism period
5. **Who was the mathematician credited with developing a theory of trigonometry and evaluating π with high accuracy?**
   a. Aryabhata
   b. Bhaskara II
   c. Brahmagupta
   d. Varahamihira
6. **In which century was the Bakhshali manuscript discovered, revealing Indian mathematicians' comfort with fractions and algebraic manipulations?**

   a. the 5th century
   b. the 7th century
   c. the 12th century
   d. the 14th century

7. **Who was the mathematician who gave zero the status of a number?**
   a. Aryabhata
   b. Bhaskara II
   c. Brahmagupta
   d. Varahamihira
8. **Which 14th-century scholar made significant advances in analysis, including producing infinite series expansions, and discovering the binomial theorem?**
   a. Aryabhata
   b. Bhaskara II
   c. Madhava
   d. Varahamihira
9. **When did the decimal system spread to the Islamic world and eventually reach Europe?**
   a. the 7th century
   b. the 9th century
   c. the 12th century
   d. the 14th century
10. **What role did Arab scholars play in the transmission of Indian mathematical knowledge to the West?**
   a. They ignored Indian contributions.
   b. They actively preserved and translated Indian texts.
   c. They opposed the decimal system.
   d. They developed their own numerical system.

Activity 186. Read the article. Review your answers to the quiz in Activity 185.

The entire course of Western mathematics was profoundly affected by a single Indian invention, that of the place-value decimal system. That every possible number can be expressed via a set of just 10 symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, by making careful use of the place of each symbol, seems such a simple notion nowadays that it is hard to appreciate its profound importance. This elegant notation system provided the means for Indian scholars to perform complicated arithmetical computations with relative ease, which in turn led to significant developments in numerical techniques, approximation methods, and the theory of arithmetic. Only when other cultures adopted the place-value decimal system from India could they accomplish the same mathematical feats that this culture had already developed.

The earliest dated evidence of mathematical activity in the Indian subcontinent goes back to the Indus civilization of 2500 B.C.E. Bronze weights and graded rods (rulers) from the period show that these people were already working with a decimal system. The Indus people worked with a basic unit of length 1.32 in. long (today called the "Indus inch"), 10 of which make their version of a "foot." Excavations show that the weights and graded rods were used extensively in construction.

The earliest written records of Indian culture are the religious texts the Vedas, dating between 1500 B.C.E. and 800 B.C.E. Although not mathematical in content, appendices to the texts give specific rules for constructing altars, exhibiting a thorough understanding of the basic principles of geometry. Early versions of the digits 0 through 9 were used at this time.

By 600 C.E., the Vedic religion had gone into decline, and Jainism came to the fore. During this period mathematics was driven by the needs of the religion and its demands for careful astronomical observations to pinpoint the exact times of religious observances and the development of an accurate calendar. The decimal representation system was now fully developed, and scholars were able to make precise and surprisingly accurate calculations. The mathematician Aryabhata (ca. 500 C.E.), for instance, had developed a theory of trigonometry to aid astronomical calculations, had developed methods for extracting square roots, evaluated π to a high degree of accuracy, and was able to find integer solutions to a large class of equations that arose from astronomical theories.

One written text from this period was discovered in 1881 in the town of Bakhshali, now in Pakistan. Written on birch bark, the Bakhshali manuscript shows that mathematicians were also comfortable with fractions, basic algebraic manipulations (they used a dot to represent an unknown quantity), and sophisticated approximation formulae.

Two mathematical research centres were formed in India during the era of Jainism, both astronomical observatories. Aryabhata headed the first centre at Kusumapura in the northeast of the Indian subcontinent, and the mathematician Varahamihira, who also made contributions to astronomy and trigonometry, headed the second centre at Ujjain, also in the north.

Varahamihira was succeeded by the 7th-century mathematician Brahmagupta, who, in his famous work "The Opening of the Universe", introduced and explained the arithmetic of non-positive numbers. He was the first mathematician in history to give zero the status of a number, defining it to be the result of subtracting a quantity from itself. Brahmagupta's work also includes a formula for the area of a cyclic quadrilateral in terms of its sides (today called Brahmagupta's formula), and presents methods for solving linear and quadratic equations, as well as systems of equations. Brahmagupta also developed sophisticated interpolation techniques for computing sine values in trigonometry.

For the next 200 years, Indian scholars worked to refine further methods of trigonometry and techniques of astronomical calculation. The mathematician Bhaskara II (Bhaskaracharya) of the 12th century made advances in number theory, algebra, combinatorics, and astronomy, and wrote a comprehensive text summarizing the state of mathematics and astronomy in India at his time. Soon afterward, other Indian scholars developed these ideas further. Jaina mathematicians also clarified the standard exponent rules and manipulated exponents in a manner that suggests today that they were also familiar with the basic principles of logarithms.

The 14th-century scholar Madhava of Sangamagramma made significant advances in analysis. He produced the infinite series expansions of trigonometric and inverse trigonometric functions (today called Taylor series), discovered the binomial theorem, and even produced Gregory's series for π, which he used to approximate its value to a considerably accurate degree.

During the first millennium India had very little contact with the cultures of the West. News of the decimal representation system, however, did manage to spread to other countries relatively quickly. A manuscript written in Syria in 662 discusses the new method of calculation, and there is evidence that the decimal system was being used in Cambodia and other Asian countries soon afterward. By the 9th century, the decimal system was in common use in the Islamic world, and from there it was quickly transmitted to Europe. Arab scholars maintained a keen interest in the work of Indian mathematicians for the centuries that followed and took an active role in preserving and translating many Indian texts.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 187. Identify the individuals based on the descriptions from the text in Activity 186.

Aryabhata / Bhaskara II (Bhaskaracharya) / Brahmagupta / Madhava of Sangamagramma / Varahamihira

1. A 12th-century mathematician who advanced number theory, algebra, combinatorics, and astronomy, contributing to the state of Indian mathematics.
2. Mathematician and astronomer who succeeded Aryabhata, contributing to astronomy and trigonometry, and heading an astronomical observatory in Ujjain.
3. Mathematician who developed a theory of trigonometry, methods for extracting square roots, and made precise astronomical calculations.
4. Mathematician who introduced zero as a number, explained the arithmetic of non-positive numbers, and developed interpolation techniques for computing sine values.
5. Scholar who made significant advances in analysis, producing infinite series expansions and contributing to the understanding of trigonometric functions.

Activity 188. Rearrange the events in chronological order according to the text of Activity 186. Provide dates where possible.

a. A manuscript in Syria discusses the decimal system, suggesting its spread beyond India.
b. Jaina mathematicians clarify exponent rules and manipulate exponents, possibly hinting at familiarity with logarithmic principles.
c. Madhava of Sangamagramma makes significant advances in analysis, producing infinite series expansions, discovering the binomial theorem, and approximating π with Gregory's series.
d. Mathematics becomes crucial for astronomical observations and calendar development.
e. The Bakhshali manuscript is discovered in Pakistan, revealing mathematicians' comfort with fractions, algebraic manipulations, and sophisticated approximation formulae.
f. The decimal system becomes common in the Islamic world, later transmitted to Europe.
g. The Indus civilization exhibits the earliest evidence of mathematical activity, using a decimal system with bronze weights and graded rods.
h. The Vedas, the earliest written records of Indian culture, provide appendices with geometric principles, showcasing early versions of the digits 0 through 9.

Activity 189. Determine whether the statements are true or false by quoting from the text in Activity 186.

1. Although not directly mathematical, the Vedas, the earliest written records of Indian culture, include guidelines for constructing altars that reveal a deep understanding of fundamental geometric principles.
2. Brahmagupta, a mathematician from the 7th century, established zero as a number, explored non-positive numbers, and presented formulas for areas and solving equations.
3. Jainism had no impact on the progress of mathematics in ancient India, and scholars primarily concentrated on religious practices rather than mathematical developments.
4. The decimal system's origins can be traced back to the Vedic period, and the Vedas extensively cover mathematical content.
5. The decimal system's usage can be traced back to the Indus civilization around 2500 B.C.E., as suggested by their utilization of a decimal system in bronze weights and graded rods.

6. The decline of Jainism and the emergence of Vedic religion prompted mathematical developments in India, particularly in accurate astronomical observations and calendar construction.
7. The dissemination of the decimal system to other nations did not occur until the 17th century.
8. The Western mathematical tradition was significantly impacted by the Indian invention of the place-value decimal system, enabling the representation of any number with just 10 symbols.

Activity 190. In groups, discuss the points. Refer to the text in Activity 186.

1. Explore the profound impact of the Indian invention of the place-value decimal system on the entire course of Western mathematics.
2. Discuss the evidence of mathematical activity in the Indus civilization, including the use of a decimal system and its application in construction.
3. Analyze the mathematical content in the appendices of the Vedas, showcasing rules for constructing altars and a grasp of basic geometric principles.
4. Examine how the decline of Vedic religion and the rise of Jainism influenced mathematics, driven by astronomical needs, resulting in precise calculations and advancements by mathematician Aryabhata.
5. Explore the significance of the Bakhshali manuscript, revealing mathematicians' comfort with fractions, algebraic manipulations, and approximation formulae.
6. Investigate the establishment of mathematical research centres, particularly astronomical observatories, during the era of Jainism, led by scholars like Aryabhata and Varahamihira.
7. Discuss Brahmagupta's contributions, including introducing zero as a number, developing interpolation techniques, and presenting formulas for solving equations.
8. Explore the 12th-century contributions of Bhaskara II in trigonometry, number theory, algebra, and astronomy, as well as the subsequent development of these ideas by other Indian scholars.
9. Examine the 14th-century scholar Madhava's significant contributions to analysis, including infinite series expansions, the binomial theorem, and approximations for π.
10. Trace the spread of the decimal system beyond India, its mention in a Syrian manuscript in 662, and its adoption in Cambodia and other Asian countries before becoming common in the Islamic world and eventually reaching Europe. Discuss the role of Arab scholars in preserving and translating Indian mathematical texts.

Activity 191. Write an overview of the text in Activity 186 using Appendix II.

Unit 25. Arabic Mathematics

Activity 192. Do the quiz on Arabic mathematics. In pairs, compare your answers.

1. **Who established the House of Wisdom in Baghdad in the 8th century?**
   a. Archimedes
   b. Caliph al-Ma’mun
   c. Euclid
   d. Muhammad ibn Musa al-Khwarizmi
2. **What was the House of Wisdom known for?**
   a. destruction of mathematical texts
   b. repository of important academic texts
   c. promoting Greek philosophy
   d. developing modern algebra
3. **Which Greek mathematician's work had a tremendous impact on Arab scholars at the House of Wisdom?**
   a. Archimedes
   b. Diophantus
   c. Euclid
   d. Menelaus of Alexandria
4. **What did al-Khwarizmi's work "Calculation by Restoration and Reduction" contribute to mathematics?**
   a. number theory
   b. algebra
   c. geometry
   d. trigonometry
5. **What departure did al-Khwarizmi make from Greek thinking?**
   a. introduction of geometry
   b. unification of arithmetic and geometry through algebra
   c. focus on trigonometry
   d. disregard for numerical representation
6. **Who refined approaches for reducing geometric problems to algebraic ones?**
   a. al-Khwarizmi
   b. al-Ma’mun
   c. al-Mahani
   d. Omar Khayyam

7. **What did Omar Khayyam attempt to develop methods for?**
   a. solving quadratic equations
   b. solving cubic equations
   c. calculating trigonometric tables
   d. generating amicable numbers
8. **Which mathematician attempted to classify all even perfect numbers?**
   a. al-Farisi
   b. al-Haytham
   c. Ibrahim ibn Sinan
   d. Thabit ibn Qurra
9. **What did Thabit ibn Qurra find a method for generating?**
   a. Fibonacci sequence
   b. prime numbers
   c. amicable numbers
   d. Pascal's triangle
10. **What did scholars in the Islamic Golden Age contribute to numeric computations?**
   a. advancements in calculus
   b. effective methods for decimal place-value representation
   c. development of the quadratic formula
   d. exploration of non-Euclidean geometries
11. **Which scholar developed effective methods for extracting the n-th root of a number?**
   a. al-Farisi
   b. al-Haytham
   c. Jamshid al-Kashi
   d. Omar Khayyam
12. **What did Ibrahim ibn Sinan introduce a method of?**
   a. differentiation
   b. integration
   c. prime factorization
   d. trigonometry

Activity 193. Read the article. Review your answers to the quiz in Activity 192.

Mathematical historians of today are grateful to the Arabic scholars of the past for preserving, translating, and honouring the great Indian, Greek, and Islamic mathematical works of the scholars before them, and for their own significant contributions to the development of mathematics. At the end of the 8th century, with the great Library of Alexandria destroyed, Caliph al-Ma’mun set up a House of Wisdom in Baghdad, Iraq, which became the next prominent centre of learning and research, as well as the repository of important academic texts. Many scholars were employed by the caliph to translate the mathematical works of the past and develop further the ideas they contained. As the Islamic empire grew over the following seven centuries, the culture of intellectual pursuit also spread. Many scholars of 12th-century Europe, and later, visited the Islamic libraries of Spain to read the texts of the Arabic academics and to learn of the advances that had occurred in the East during the dark ages of the West. A significant amount of mathematical material was transmitted to Europe via these means.

One of the first Greek texts to be translated at the House of Wisdom was Euclid's famous treatise, "The Elements". This work made a tremendous impact on the Arab scholars of the period, and many of them, when conducting their own research, formulated theorems, and proved results precisely in the style of Euclid. Members of the House of Wisdom also translated the works of Archimedes of Syracuse, Diophantus of Alexandria, Menelaus of Alexandria, and others, and so they were certainly familiar with all the great Greek advances in the topics of geometry, number theory, mechanics, and analysis. They also translated the works of Indian scholars, Aryabhata and Bhaskara, for instance, and were familiar with the theory of trigonometry, methods in astronomy, and further topics in geometry and number theory. Any Arab scholar who visited the House of Wisdom had, essentially, the entire bulk of human mathematical knowledge available to him in his own language.

The Arab mathematician Muhammad ibn Musa al-Khwarizmi (ca. 800) wrote a number of original texts that were enormously influential. His first piece simply described the decimal place-value system he had learned from Indian sources. Three hundred years later, when translated into Latin, this work became the primary source for Europeans who wanted to learn the new system for writing and manipulating numbers. But more important was al-Khwarizmi's piece "Calculation by Restoration and Reduction", from which the topic of algebra arose. Al-Khwarizmi was fortunate to have all sources of mathematical knowledge available to him. He began to see that the then-disparate notions of "number" and "geometric magnitude" could be unified as one whole by developing the concept of algebraic objects. This represented a significant departure from Greek thinking, in which mathematics is synonymous with geometry. Al-Khwarizmi's insight provided a means to study both arithmetic and geometry under a single framework, and his methods of algebra paved the way for significant developments in mathematics of much broader scope than ever previously envisioned.

The mathematician al-Mahani (ca. 820) developed refined approaches for reducing geometric problems to algebraic ones. He showed, in particular, that the famous problem of duplicating the cube is essentially an algebraic issue. Other scholars brought rigour to the subject by proving that certain popular, but complicated, algebraic methods were valid. These scholars were comfortable manipulating polynomials and developed rules for working with exponents. They solved linear and quadratic equations, as well as various systems of equations. Surprisingly, no one of the time thought to ease matters by using symbols to represent quantities: all equations and all manipulations were described fully in words each and every time they were employed.

With quadratic equations well understood, the scholar Omar Khayyam (ca. 1048–1131) attempted to develop methods of solving degree-three equations. Although he was unable to develop general algebraic methods for this task, he did find ingenious geometric techniques for solving certain types of cubics with the aid of conic sections. He was aware that such equations could have more than one solution.

In number theory, Thabit ibn Qurra (ca. 836–901) found a beautiful method for generating amicable numbers. This technique was later utilized by al-Farisi (ca. 1260–1320) to yield the pair 17,296 and 18,416, which today is usually attributed to Leonhard Euler (1707–83). In his writing, Omar Khayyam referred to earlier Arab texts, now lost, that discuss the equivalent of Pascal's triangle and its connections to the binomial theorem. The mathematician al-Haytham (ca. 965–1040) attempted to classify all even perfect numbers.

Taking advantage of the ease of the Indian system of decimal place-value representation, Arabic scholars also made great advances in numeric computations. The great 14th-century scholar Jamshid al-Kashi developed effective methods for extracting the n-th root of a number and evaluated π to a significant number of decimal places. Scholars at the time also developed effective methods for computing trigonometric tables and techniques for making highly accurate computations for the purposes of astronomy.

On a theoretical note, scholars also advanced the general understanding of trigonometry and explored problems in spherical geometry. They also investigated the philosophical underpinnings of geometry, focusing, in particular, on the role the famous parallel postulate plays in the theory. Omar Khayyam, for instance, attempted to prove the parallel postulate — failing, of course — but did accidentally prove results about figures in non-Euclidean geometries along the way. The mathematician Ibrahim ibn Sinan (908–946) also introduced a method of "integration" for calculating volumes and areas following an approach more general than that developed by Archimedes of Syracuse (ca. 287–212 B.C.E.). He also applied his approach to the study of conic sections and to optics.

(by James Tanton, from Encyclopedia of Mathematics)

Activity 194. Identify the individuals based on the descriptions from the text in Activity 193.

al-Farisi / al-Haytham / al-Mahani / Archimedes of Syracuse / Aryabhata / Bhaskara / Diophantus of Alexandria / Ibrahim ibn Sinan / Jamshid al-Kashi / Menelaus of Alexandria / Muhammad ibn Musa al-Khwarizmi / Omar Khayyam / Thabit ibn Qurra

1. An Arab mathematician who introduced a method of "integration" for calculating volumes and areas, with applications to conic sections and optics.
2. Arab mathematician who utilized Thabit ibn Qurra's method to find an amicable number pair.
3. Arab mathematician who attempted to classify even perfect numbers and made contributions to various mathematical fields.
4. Arab mathematician of the 14th century who developed effective methods for extracting roots and calculated π to many decimal places.
5. Arab mathematician who attempted to develop methods for solving cubic equations and made contributions to number theory.
6. Arab mathematician who developed refined approaches for reducing geometric problems to algebraic ones.
7. Arab mathematician who found a method for generating amicable numbers in number theory.
8. Arab mathematician who wrote influential texts on the decimal place-value system and algebra.
9. Greek mathematicians whose works were translated at the House of Wisdom, contributing to Arab scholars' knowledge.
10. Indian scholars whose works were translated at the House of Wisdom, introducing trigonometry and astronomy to Arab scholars.

Activity 195. Rearrange the events in chronological order according to the text of Activity 193. Provide dates where possible.

a. Al-Mahani develops refined approaches for reducing geometric problems to algebraic ones, highlighting the algebraic nature of the famous problem of duplicating the cube.

b. Ibrahim ibn Sinan introduces a method of "integration" for calculating volumes and areas.
c. Islamic libraries in Spain attract European scholars.
d. Jamshid al-Kashi advances numeric computations, extracting the n-th root of a number and calculating π to a significant number of decimal places.
e. Muhammad ibn Musa al-Khwarizmi writes influential texts, describing the decimal place-value system and introducing algebra through "Calculation by Restoration and Reduction."
f. Omar Khayyam attempts to develop methods for solving degree-three equations and introduces ingenious geometric techniques for certain cubics using conic sections.
g. Omar Khayyam attempts to prove the parallel postulate, unintentionally contributing to non-Euclidean geometries.
h. Thabit ibn Qurra discovers a method for generating amicable numbers.

Activity 196. Determine whether the statements are true or false by quoting from the text in Activity 193.

1. "The Elements" by Euclid was among the initial Greek texts translated at the House of Wisdom, leaving a notable impact on Arab scholars.
2. Al-Khwarizmi's contribution to the decimal place-value system became a key source for Europeans when translated into Latin.
3. Al-Khwarizmi's insight significantly contributed to the unification of algebra and geometry under a single framework.
4. Al-Mahani developed and improved methods for converting algebraic problems into geometric forms.
5. Arabic scholars described equations and manipulations using algebraic notation.
6. Contemporary mathematical historians acknowledge the significance of ancient Arabic scholars in preserving and translating mathematical works from diverse traditions.
7. Following the destruction of the Library of Alexandria, Caliph al-Ma’mun founded the House of Wisdom in Baghdad, which emerged as a significant centre for learning.
8. Omar Khayyam was unsuccessful in developing general algebraic methods for solving degree-three equations.
9. Thabit ibn Qurra's method for generating amicable numbers had a notable influence on later mathematicians.

Activity 197. In groups, discuss the points. Refer to the text in Activity 193.

1. Discuss the role of the House of Wisdom in Baghdad, Iraq, and its significance as a centre for preserving, translating, and developing mathematical knowledge from

Indian, Greek, and Islamic traditions. How did the destruction of the Library of Alexandria contribute to the establishment of such centres?

2. Explore the impact of translating Greek texts, including Euclid's "The Elements," at the House of Wisdom. How did this influence Arab scholars and contribute to the development of geometry, number theory, mechanics, and analysis in the Islamic world?
3. Examine the contributions of Muhammad ibn Musa al-Khwarizmi to the field of mathematics, particularly in the context of his work on the decimal place-value system and the emergence of algebra. How did al-Khwarizmi's insights represent a departure from Greek mathematical thinking?
4. Analyze the development of algebraic methods by al-Mahani and other scholars, including their approaches to reducing geometric problems to algebraic ones. Discuss the significance of their contributions to solving equations and manipulating polynomials.
5. Explore Omar Khayyam's attempt to develop methods for solving degree-three equations and his ingenious geometric techniques for certain types of cubics using conic sections. How did Khayyam's work contribute to the understanding of cubic equations?
6. Discuss the method for generating amicable numbers found by Thabit ibn Qurra and later utilized by al-Farisi. How did this technique contribute to the study of number theory, and what are the connections to later mathematicians like Leonhard Euler?
7. Explore the advancements made by Arabic scholars in numeric computations, including the effective methods for extracting the n-th root of a number by Jamshid al-Kashi. How did these developments contribute to practical applications and mathematical accuracy?
8. Discuss the theoretical advancements in trigonometry and spherical geometry by Islamic scholars. How did their exploration of the parallel postulate and attempts to prove it contribute to the understanding of non-Euclidean geometries?
9. Explore Ibrahim ibn Sinan's method of "integration" for calculating volumes and areas, and its generality compared to Archimedes' approach. How did this contribute to the theoretical understanding of mathematical concepts beyond what was known in earlier times?
10. Discuss the interdisciplinary nature of mathematical exploration in the Islamic world, incorporating optics and other fields. How did this interdisciplinary approach contribute to the overall development of mathematics and its applications in various domains?

Activity 198. Write an overview of the text in Activity 193 using Appendix II.

“Nature is written in mathematical language.” (Galileo Galilei)