

Activity 105. Read the article to complete the table differentiating between the three geometries.

After numerous unsuccessful attempts throughout history to establish the parallel postulate as a consequence of the remaining four of Euclid's postulates, mathematicians began to contemplate theories of geometry in which the fifth postulate does not hold. Any such theory of geometry is called a non-Euclidean geometry.

In 1795 the Scottish mathematician and physicist John Playfair (1748–1819) presented an alternative, but equivalent, formulation of the parallel postulate:

Through any point in the plane, there is precisely one line through that point parallel to any prescribed direction.

Recasting the postulate this way makes it apparent that negation of the famous fifth postulate has two parts. Either:

1. *There are no lines through a given point parallel to a given direction.*
2. *There is more than one line through a given point parallel to a given direction.*

Independently discovered in 1823 by the Hungarian mathematician János Bolyai (1802–60) and in 1829 by the Russian mathematician Nikolai Ivanovich Lobachevsky (1792–1856), hyperbolic geometry (Lobachevskian geometry) is a non-Euclidean geometry in which the famous parallel postulate fails in the following manner:

Through a given point not on a given line, there is more than one line parallel to that given line.

The French mathematician Jules-Henri Poincaré (1854–1912) later provided a simple model for this geometry and the means to easily visualize geometric results in this theory. The “Poincaré disk” consists of all the points in the interior of the unit circle. A “point” in hyperbolic geometry is any point inside this circle, and a “line” is to be interpreted as a circular arc within the circle with endpoints perpendicular to the boundary of the circle. Any diameter of the boundary circle is also considered a line. Distances are not measured with a traditional ruler: points on the boundary circle are considered to be infinitely far from the centre of the circle.

Bolyai and Lobachevsky showed that all but the fifth of Euclid's postulates hold in the hyperbolic geometry and, moreover, that this model of geometry is consistent (that is, free of contradictions). This establishes that the parallel postulate cannot be logically deduced as a consequence of the remaining axioms proposed by Euclid.

In hyperbolic geometry, all angles in triangles sum to less than 180° , and the ratio of the circumference of any circle to its diameter is less than π . (Moreover, the value of this ratio is not the same for all circles.) Also, it is possible for two perpendicular lines to be parallel to the same line.

Physicists, following the work of Albert Einstein, suggest that the geometry of our universe is hyperbolic: that it appears to us as Euclidean is a result of the fact that we occupy such a small portion of it. (This is analogous to the fact that it is difficult to recognize the Earth as round when living on it.)

Discovered in 1856 by the German mathematician Georg Friedrich Bernhard Riemann (1826–66), and later slightly modified by Felix Klein (1849–1925), elliptic geometry (Riemannian geometry) is a non-Euclidean geometry in which the famous parallel postulate fails in the following manner:

Through a given point not on a given line, there are no lines parallel to that given line.

Riemann used the surface of a sphere as a model of this geometry by interpreting the word “line” to mean a great circle on the sphere. Given that in a theory of geometry two lines are meant to intersect at just one point (yet any two great circles intersect at two antipodal points), it is appropriate then to interpret the word “point” in elliptic geometry as an antipodal pair of points on the surface. In this setting, it is now also true that any two distinct points determine a unique line.

Riemann and Klein proved that all but the fifth of Euclid’s postulates hold in this model and, moreover, that this model is consistent (that is, free of contradictions). This establishes that the parallel postulate cannot be logically deduced as a consequence of the remaining axioms proposed by Euclid.

In elliptic geometry all angles in triangles sum to more than 180° , and the ratio of the circumference of any circle to its diameter is greater than π (and this value varies from circle to circle).

(from Elementary Geometry for College Students)