

Activity 122. Read the article. Review your answers to the quiz in Activity 121.

From very early times, surveyors, architects, navigators, and astronomers have made use of triangles to determine distances that could not be measured directly. This gave birth to the subject we today know as trigonometry. There are problems in the ancient Egyptian text, the Rhind papyrus of around 1650 B.C.E., that call for the determination of the slope angles of pyramid faces using the equivalent of the cotangent function we use today, and a Babylonian clay tablet from around 1700 B.C.E. contains a table of secant values for the 15 angles between 30° and 45° . The Greek philosopher Thales of Miletus (ca. 600 B.C.E.) is said to have made use of similar triangles to determine the height of the Cheops pyramid by comparing the length of its shadow with the length of the shadow of a rod inserted in the ground. Eratosthenes of Cyrene (ca. 250 B.C.E.) computed the circumference of the Earth using lengths of shadows and a simple geometric argument on angles.

Greek astronomers of ancient times believed that the stars and planets of the night sky moved along circular arcs of a giant celestial sphere, and they worked to develop models that would accurately predict the motion of these objects on the sphere. Rather than phrase matters in terms of angles, which proved to be difficult, Greek astronomers chose to work with measures of straight lengths closely related to angles, namely, the lengths of chords of circles.

Hipparchus of Rhodes (ca. 200 B.C.E.) constructed a table of such chord lengths for a circle of circumference $21,600 = 360 \times 60$ units (which corresponds to 1 unit of circumference for each minute of arc).

In the 2nd century C.E., the mathematician Claudius Ptolemy wrote the first extensive treatise on the theory of chords and their use in obtaining information about “spherical triangles,” that is, triangles made by great circular arcs on the surface of a sphere. In addition to working out theorems, Ptolemy explained how to construct tables of chord values, and presented his own list of chord values for all angles between zero and 180° in half-degree increments.

The next important step in the development of trigonometry occurred in India. Scholars of the 5th century C.E. had by this time discovered that working with half-chords for half-angles greatly simplified the theory of chords and its applications to astronomy. This approach is almost the same construct as the sine function of today. Whereas we think of sine as a ratio of lengths (the length of the half-chord to the radius), Indian scholars interpreted sine as the actual length of the half-chord. Of course, this value of an angle differed for circles of different sizes. The scholars Aryabhata, Bhaskara II, and others developed astonishingly sophisticated techniques for computing half-chord values.

The Arab scholars of the 10th century took a great interest in the work from India. The mathematician Abu al-Wafa (ca. 950) of Baghdad systemized theorems and proofs of Indian trigonometry and prepared his own comprehensive table of half-chord values. He is also

believed to have invented the tangent function, which he called the “shadow,” and possibly the secant and the cosecant functions. (Still, all were thought of as specific lengths, not as ratios of lengths.)

In the 12th century, European scholars began translating the Arabic works and soon became acquainted with the extensive theory of trigonometry. The famous scholar Fibonacci (ca 1170–1250) also travelled extensively in the Arab countries and wrote of their trigonometry in his famous work “Practica geometriae”. Around 1464, the German astronomer and mathematician Regiomontanus (1436–76) compiled “De triangulis omnimodis”, a compendium of trigonometry of that time. This work was enormously influential, and over the following decades other texts on the topic appeared. The German mathematician Georg Joachim Rheticus (1514–74) published, in 1551, “Canon doctrinae triangulorum”, which defined, for the first time, all six basic trigonometric functions, and explained how to relate them to right triangles without making any reference to circles. The Danish physician Thomas Fincke (1561–1656) coined the terms “tangent” and “secant” and developed further fundamental trigonometric formulae. The word “trigonometry” itself was invented by the German mathematician Bartholomaeus Pitiscus (1561–1613). By this time, trigonometry was certainly regarded as a worthwhile topic of mathematical pursuit, independent of applications to astronomy.

The subject also proved to be useful in the study of algebra. The French mathematician François Viète (1540–1603) showed, for example, that one could solve certain cubic equations by making trigonometric substitutions. His famous formula for π can be derived with repeated use of trigonometric functions.

Up until this point, sine values, as well as the other trigonometric values, were still regarded as actual line lengths and not ratios of lengths. After the invention of calculus, Leonhard Euler (1707–83) came to realize that it is appropriate to think of sine not as a physical length, but rather as a function of angle independent of length. He suggested that this could be accomplished by scaling all circles under consideration to unit circles, an operation that is equivalent to dividing all quantities by the radius of the circle. Thus, for the first time, all trigonometric quantities came to be thought of as ratios. In 1748 Euler wrote “Introductio in analysin infinitorum”, which became the dominating textbook on the topic of trigonometry for the century that followed. It essentially outlines the principles of trigonometry as we regard them today.

(by James Tanton, from Encyclopedia of Mathematics)