Activity 150. Read the article. Review your answers to the quiz in Activity 149.

The study of calculus begins with the study of motion, a topic that has fascinated and befuddled scholars since the time of antiquity. The first recorded work of note in this direction dates back to the Greek scholars Pythagoras (ca. 569–475 B.C.E) and Zeno of Elea (ca. 500 B.C.E.), and their followers, who put forward the notion of an infinitesimal as one possible means for explaining the nature of physical change. Motion could thus possibly be understood as the aggregate effect of a collection of infinitely small changes. Zeno, however, was very much aware of fundamental difficulties with this approach and its assumption that space and time are consequently each continuous and thus infinitely divisible. Through a series of ingenious logical arguments, Zeno reasoned that this cannot be the case. At the same time, Zeno presented convincing reasoning to show that the reverse position, that space is composed of fundamental indivisible units, also cannot hold. The contradictory issues proposed by Zeno were not properly resolved for well over two millennia.

The concept of the infinitesimal also arose in the ancient Greek study of area and volume. Scholars of the schools of Plato (428–348 B.C.E.) and of Eudoxus of Cnidus (ca. 370 B.C.E.) developed a "method of exhaustion," which attempted to compute the area or volume of a curved figure by confining it between two known quantities, both of which can be made to resemble the desired object with any prescribed degree of accuracy. Archimedes of Syracuse (287–212 B.C.E.) applied this method to compute the area of a section of a parabola, and 600 years later, Pappus of Alexandria (ca. 300–350 C.E.) computed the volume of a solid of revolution via this technique. Although successful in computing the areas and volumes of a select collection of geometric objects, scholars had no general techniques that allowed for the development of a general theory of area and volume. Each individual calculation for a single specific example was hailed as a great achievement in its own right.

The resurgence of scientific investigation in the mid-1600s led European scholars to push the method of exhaustion beyond the point where Archimedes and Pappus had left it. Johannes Kepler (1571–1630) extended the use of infinitesimals to solve optimization problems. Others worked on the problem of finding tangents to curves, an important practical problem, and the problem of finding areas of irregular figures. In 1635, the Italian mathematician Bonaventura Cavalieri wrote the first textbook on what we would call "integration methods". He described a general "method of indivisibles" useful for computing volumes. The principle today is called Cavalieri's principle.

The French mathematician Gilles Personne de Roberval (1602–75) was the first to link the study of motion to geometry. He realized that the tangent line to a geometric curve could be interpreted as the instantaneous direction of motion of a point travelling along that curve. The philosopher and mathematician René Descartes (1596–1650) developed general techniques for finding the formula for the tangent line to a curve at a given point. This technique was later picked up by Pierre de Fermat (1601–65), who used the study of tangents to solve maxima and minima problems in much the same way we solve such problems today. As a separate area of study, Fermat also developed techniques of integral calculus to find areas between curves and lengths of arcs of curves, which were later developed further by Blaise Pascal (1623–62) and English mathematicians John Wallis (1616–1703) and Isaac Barrow (1630–77).

At the same time scholars, including Wallis, began studying series and infinite products. The Scottish mathematician James Gregory (1638–75) developed techniques for expressing trigonometric functions as infinite sums, thereby discovering Taylor series 40 years before Brook Taylor (1685–1731) independently developed the same results.

By the mid-1600s, certainly, all the pieces of calculus were in place. Yet scholars at the time did not realize that all the varied problems being studied belonged to one unified whole, namely, that the techniques used to solve tangent problems could be used to solve area problems, and vice versa. A fundamental breakthrough came in the 1670s when, independently, Gottfried Wilhelm Leibniz (1646–1716) of Germany and Sir Isaac Newton (1642–1727) of England discovered an inverse relationship between the “tangent problem” and the “area problem.” The discovery of the fundamental theorem of calculus brought together the disparate topics being studied, provided a beautiful and natural perspective on the subject as a whole, and allowed scholars to make significant advances in solving geometric and physical problems with spectacular success. Despite the content of knowledge that had been established up until that time, it is the discovery of the fundamental theorem of calculus that represents the discovery of calculus.

Newton approached calculus through a concept of “flowing entities.” He called any quantity being studied a “fluent” and its rate of change a “fluxion”. Records show that he had developed these ideas as early as 1665, but he did not publish an account of his theory until 1704. Unfortunately, his writing style and choice of notation also made his version of calculus accessible only to a select audience. Leibniz, on the other hand, made explicit use of an infinitesimal in his development of the theory. He called the infinitesimal change of a quantity “x” a differential, denoted “dx”. Leibniz invented a beautiful notational system for the subject that made reading and working with his account of the theory immediately accessible to a wide audience. (Many of the symbols we use today in differential and integral calculus are due to Leibniz.) Leibniz formulated his approach in the mid-1670s and published his account of the subject in 1684. Although it is now known that Newton and Leibniz had made their discoveries independently, matters at the time were not clear, and a bitter dispute arose over the priority for the discovery of calculus.

Applying the techniques to problems of the real world became the main theme of 18^{th}-century mathematics. Newton’s famous 1687 text “Principia” paved the way with its analysis of the laws of motion and the mechanics of the solar system. The Swiss brothers Jakob Bernoulli (1654–1705) and Johann Bernoulli (1667–1748) of the famous Bernoulli family, champions of Leibniz in the famous dispute, studied the newly invented calculus and were the first to give public lectures on the topic. Johann Bernoulli was hired to teach differential calculus to the French nobleman Guillaume François de L'Hopital (1661–1704) via written correspondence. In 1696 L'Hopital then published the content of Johann's letters with his own name as author. The Italian mathematician Marie Gaetana Agnesi (1718–99) wrote the first comprehensive textbook dealing with both differential and integral calculus in 1755.

The Swiss mathematician Leonhard Euler (1707–83) and French mathematicians Joseph-Louis Lagrange (1736–1813) and Pierre-Simon Laplace (1749–1827) were prominent in developing the theory of differential equations. Euler also wrote extensively on the subject of calculus, showing how the theory can be applied to a vast range of pure and applied mathematical problems. Yet despite the evident success of calculus, some 18th-century scholars questioned the validity and the soundness of the subject.

The sharpest critic of Newton's and Leibniz's work was the Anglican Bishop of Coyne, George Berkeley (1685–1753). In his scathing essay, "The Analyst," Berkeley demonstrated, convincingly, that both Newton's notion of a fluxion and Leibniz's concept of an infinitesimal are ill-defined, and that the foundations of the subject are consequently insecure. Mathematicians consequently began looking for ways to put calculus on a sound footing. Significant progress was not made until the 19th century, when the French mathematician Augustine Louis Cauchy (1789–1857) suggested that the notion of an infinitesimal should be replaced by that of a limit. German mathematician Karl Weierstrass (1815–97) developed this idea further and was the first to give absolutely clear and precise definitions to all concepts used in calculus, devoid of any mystery or reliance on geometric intuition. The work of the German mathematician Richard Dedekind (1831–1916) highlighted the role properties of the real number system play in ensuring the validity of the intermediate-value theorem and extreme-value theorem and all the essential results that follow from them.

Initially, calculus was deemed a theory pertaining only to continuous change and continuous functions. The German mathematician Bernhard Riemann (1826–66) was the first to consider, and give careful discussion on, the integration of discontinuous functions. His definition of an integral is the one typically presented in textbooks today. At the end of the 19th century, the French mathematician Henri Léon Lebesgue (1875–1941) literally turned Riemann's approach around and developed a concept of integration that can be applied to a much wider class of functions and class of settings. In order to do this, Lebesgue had to develop a general "measure theory" for determining the size of complicated sets. His new theory proved to be fundamentally important, and it now has profound applications to a wide range of mathematical topics. It proved to be especially important to the sound development of probability theory.

(by James Tanton, from Encyclopedia of Mathematics)