

Activity 69. Read the article to complete the table based on the text.

The development of different types of numbers can be seen as motivated by the need for solving different types of equations. For example, the counting numbers (that is, the natural numbers N) suffice for solving any equation of the type $x + 2 = 5$, for instance, but not an equation of the type $x + 5 = 2$. (There is no solution to this equation within the set of counting numbers.) This motivates the introduction of negative numbers and the construction of the integers Z (from the German word “Zahlen” for “numbers”).

Working solely in the realm of the whole numbers, a number is said to be even if it is divisible by 2, and odd if it leaves a remainder of 1 when divided by 2. For example, 18 is divisible by 2 and so is even, and 23 leaves a remainder of 1 and so is odd. As the study of evenness and oddness shows, the number 0 is even. Two integers that are either both even or both odd are said to have the same parity. For instance, 17 and 53 have the same parity (both are odd), and 9 and 14 have opposite parity. Sometimes the term “parity” is used in a more general setting as to mean “being in one of two possible states” (either positive or negative).

A whole number possessing just two positive factors is called a prime number, or simply a prime. For example, 7 has only two positive factors, namely 1 and 7, and so is prime. The number 24 has eight positive factors and so is not prime, and the number 1 has only one factor and is not prime. The term composite is used to describe numbers greater than 1 that are not prime, that is positive whole numbers with more than two positive factors. (Medieval mathematician Fibonacci (1170–1250) called prime numbers “incomposite.”) It is vital that the number 1 be considered neither prime nor composite for the fundamental theorem of arithmetic to hold true.

But the set of integers is not always sufficient for solving equations of the type $5x = 3$, for instance. Desiring solutions to equations of this type leads to the construction of fractions and the set of all rational numbers Q (for “quotient”). A rational number is any number that can be written in the form a/b , where “a” and “b” represent integers and $b \neq 0$. The set of rational numbers is the set of all terminating and all repeating decimals.

Unfortunately, again, not all equations can be solved within the rational system. For example, the equation $x^2 - 2 = 0$ has no rational solution. Extending the set of rational numbers to include solutions to equations of this type introduces irrational numbers and the construction of the real number system R .

An irrational number is a nonterminating, nonrepeating decimal. An irrational number cannot be expressed as a fraction with an integer numerator and a nonzero integer denominator. Two subsets of irrationals are algebraic and transcendental numbers. A number is called algebraic if it is the root of a polynomial with integer coefficients. For example, $(1/2)(5 + \sqrt{13})$ is algebraic since it is a solution to the equation $x^2 - 5x + 3 = 0$. Numbers that are not algebraic are called transcendental.

It is extraordinarily difficult to define precisely what is meant by a real number. Many standard texts in mathematics define a real number to be any rational number or any irrational number.

A real number “ x ” is said to be positive if it is greater than zero, that is, if $x > 0$. A real number less than zero is called negative. An unspecified real number that is positive or possibly zero is called nonnegative. One that is negative or possibly zero is called nonpositive. Zero is the only real number that is neither positive nor negative.

Yet the system of real numbers also does not suffice for solving all equations. With the introduction of a single additional number, denoted “ i ”, to represent an “imaginary” solution to the equation $x^2 + 1 = 0$, the complex numbers C are born. The number “ i ” is usually regarded as the square root of negative one: $i = \sqrt{-1}$. (One must be careful as there are, in fact, two square roots of this quantity, namely “ i ” and “ $-i$ ”.) Surprisingly, as shown by the fundamental theorem of algebra, the introduction of this single number is all that is needed to solve any polynomial equation $a_n x^n + \dots + a_1 x + a_0 = 0$. Thus, the complex numbers represent a system of numbers that is algebraically closed in the sense that the construction of no new type of number is needed to solve arithmetic equations.

On a conceptual level, the notion of “number” is intimately connected with the act of counting. Simple counting systems of ancient times used tally marks to record numbers, and over the millennia this basic numeration scheme evolved to the sophisticated place-value system we use today. (The ancient Egyptians of around 3000 B.C.E. were perhaps the first to move from the use of tally marks alone.) It was a great intellectual achievement for mankind when the notion of “number” was removed from the specific objects being counted, recognizing, for instance, that two cows, two houses, and two days all share a common property of “two-ness.” (Even today we sometimes use different words to count different types of “two.” For instance, the words “twins”, “couple”, and “pair” cannot be used interchangeably to represent two people.) This simple recognition of an abstract commonality between sets of objects was exploited by German mathematician Georg Cantor (1845–1918) who, in the late 1800s, developed a general notion of cardinality. With it, Cantor extended the notion of “number” to include counts of sets of infinite size. He established, for instance, that there are an infinite number of different types of infinity and managed to develop a meaningful system of arithmetic for his transfinite numbers.

The Irish mathematician Sir William Rowan Hamilton (1805–65) followed a different route and worked to extend the notion of “number” to represent operations on n -dimensional space. An Argand diagram shows that the complex numbers have a natural representation as points on a plane. Hamilton sought to give meaning to an arithmetic for points in three- and higher-dimensional space. Although he did not succeed in accomplishing this goal for three-dimensional space, his invention of the quaternions shows this feat can be done in four-dimensional space. (The octonions provide an arithmetic for eight-dimensional space.)

(from Elementary Algebra)