Activity 82. Read the article. Review your answers to the quiz in Activity 81.

Finding solutions to equations is a pursuit that dates back to the ancient Egyptians and Babylonians and can be traced through the early Greeks' mathematics. The Rhind papyrus, dating from around 1650 B.C.E., for instance, contains a problem reading:

A quantity; its fourth is added to it. It becomes fifteen. What is the quantity?

Readers are advised to solve problems like these by a method of "false position," where one guesses (posits) a solution, likely to be wrong, and adjusts the guess according to the result obtained. In this example, to make the division straightforward, one might guess that the quantity is four. Taking 4 and adding to it its fourth gives, however, only 4 + 1 = 5, one-third of the desired answer of 15. Multiplying the guess by a factor of three gives the solution to the problem, namely 4 × 3, which is 12.

Although the method of false position works only for linear equations of the form ax = b, it can nonetheless be an effective tool. In fact, several of the problems presented in the Rhind papyrus are quite complicated and are solved relatively swiftly via this technique.

Clay tablets dating back to 1700 B.C.E. indicate that Babylonian mathematicians were capable of solving certain quadratic equations by the method of completing the square. They did not, however, have a general method of solution and worked only with a set of specific examples fully worked out. Any other problem that arose was matched with a previously solved example, and its solution was found by adjusting the numbers appropriately.

Much of the knowledge built up by the old civilizations of Egypt and Babylonia was passed on to the Greeks. They took matters in a different direction and began examining all problems geometrically by interpreting numbers as lengths of line segments and the products of two numbers as areas of rectangular regions. Followers of Pythagoras from the period 540 to 250 B.C.E., for instance, gave geometric proofs of the distributive property and the difference of two squares formula, for example, in much the same geometric way we use today to explain the method of expanding brackets. The Greeks had considerable trouble solving cubic equations, however, since their practice of treating problems geometrically led to complicated three-dimensional constructions for coping with the product of three quantities.

At this point, no symbols were used in algebraic problems, and all questions and solutions were written out in words (and illustrated in diagrams). However, in the 3rd century, Diophantus of Alexandria introduced the idea of abbreviating the statement of an equation by replacing frequently used quantities and operations with symbols as a kind of shorthand. This new focus on symbols had the subtle effect of turning Greek thinking away from geometry. Unfortunately, the idea of actually using the symbols to solve equations was ignored until the 16th century.

The Babylonian and Greek schools of thought also influenced the development of mathematics in ancient India. The scholar Brahmagupta (ca. 598–665) gave solutions to quadratic equations and outlined general methods for solving systems of equations containing several variables. (He also had a clear understanding of negative numbers and was comfortable working with zero as a valid numerical quantity.) The scholar Bhaskara (ca. 1114–85) used letters to represent unknown quantities and, in working with quadratic equations, suggested that all positive numbers have two square roots and that negative numbers have no (meaningful) roots.

A significant step toward the development of modern algebra occurred in Baghdad, Iraq, in the year 825 when the Arab mathematician Muhammad ibn Musa al-Khwarizmi (ca. 780–850) published his famous piece “Calculation by Restoration and Reduction”. This work represents the first clear and complete exposition on the art of solving linear equations by a new practice of performing the same operation on both sides of an equation. For example, the expression $x - 3 = 7$ can be “restored” to $x = 10$ by adding three to both sides of the expression, and the equation $5x = 10$ can be “reduced” to $x = 2$ by dividing both sides of the equation by five. Al-Khwarizmi also showed how to solve quadratic equations via similar techniques. His descriptions, however, used no symbols, and like the ancient Greeks, al-Khwarizmi wrote everything out in words. Nonetheless, al-Khwarizmi’s treatise was enormously influential, and his new approach to solving equations paved the way for modern algebraic thinking. In fact, it is from the word “al-jabr” in the title of his book that our word “algebra” is derived.

Al-Khwarizmi’s work was translated into Latin by the Italian mathematician Fibonacci (ca. 1175–1250), and his efficient methods for solving equations quickly spread across Europe during the 13th century. The art of algebra became known in Europe as “the cossic art” (from the Italian word “cosa” for “thing”).

Renaissance scholars Scipione del Ferro (1465–1526) and Niccolò Tartaglia (ca. 1500–57) both knew how to solve cubic equations. In 1545 Girolamo Cardano (1501–76) published “The Great Art”, which included solutions to the cubic and quartic equations discovered by his assistant Ludovico Ferrari (1522–65).

By the end of the 17th century, mathematicians were comfortable performing the same sort of symbolic manipulations we practice today and were willing to accept negative numbers and irrational quantities as solutions to equations. The French mathematician François Viète (1540–1603) introduced an efficient system for denoting powers of variables and was the first to use letters as coefficients before variables, as in $ax^2 + bx + c$, for instance. (Viète also introduced the signs + and –, although he never used a sign for equality.) René Descartes (1596–1650) introduced the convention of denoting unknown quantities by the last letters of the alphabet, “x”, “y”, and “z”, and known quantities by the first, “a”, “b”, “c”. (This convention is now completely ingrained; when we see, for example, an equation of the form $ax + b = 0$, we assume, without question, that it is for “x” we must solve.)

The German mathematician Carl Friedrich Gauss (1777–1855) proved the fundamental theorem of algebra in 1797, which states that every polynomial equation of degree “n” has at least one and at most “n” (possibly complex) roots. His work, however, does not provide actual methods for finding these roots.

For the centuries that followed, mathematicians attempted to find a general arithmetic method for solving all quintic (fifth-degree) equations. Leonhard Euler (1707–83) suspected that the task might be impossible. Between the years 1803 and 1813, the Italian mathematician Paolo Ruffini (1765–1822) published a number of algebraic results that strongly suggested the same, and just a few years later the Norwegian mathematician Niels Henrik Abel (1802–29) proved that, indeed, there is no general formula that solves all quintic equations in a finite number of arithmetic operations. Of course, some degree-five equations can be solved algebraically. (Equation of the form $x^5 - a = 0$, for instance, have solutions $\sqrt[5]{x} = \sqrt{a}$.) In 1831 the French mathematician Évariste Galois (1811–32) completely classified those equations that can be so solved, developing work that gave rise to a whole new branch of mathematics today called group theory.

In the 19th century mathematicians began using variables to represent quantities other than real numbers. For example, English mathematician George Boole (1815–64) invented an algebra of symbolic logic in which variables represented sets, and the Irish scholar Sir William Rowan Hamilton (1805–65) invented algebraic systems in which variables represented vectors or quaternions.

With these new systems, important characteristics of algebra changed. Hamilton, for instance, discovered that multiplication was no longer commutative in his systems: a product a × b might not necessarily give the same result as b × a. This motivated mathematicians to develop abstract axioms to explain the workings of different algebraic systems. Thus, the topic of abstract algebra was born. One outstanding contributor in this field was German mathematician Amalie Noether (1883–1935), who made important discoveries about the nature of noncommutative algebras.

(by James Tanton, from Encyclopedia of Mathematics)