Activity 98. Read the article to examine the enduring impact of Euclid's "The Elements" on logic, mathematics, and education.

The branch of mathematics concerned with the properties of space and of figures, lines, curves, and points drawn in space is called geometry. Plane geometry examines objects drawn in a plane (lines, circles, polygons, and the like), solid geometry, or stereometry, deals with figures in three-dimensional space (polyhedra, lines, planes, and surfaces), and spherical geometry studies the properties of lines and shapes drawn on the surface of a sphere. The word "geometry" comes from the Greek words "ge" meaning "earth" and "metria" meaning "measure." As the origin of the word suggests, the study of geometry evolved from very practical concerns with regard to the accurate measurement of tracts of land, navigation, and architecture.

The geometry based on the definitions and axioms set out in Euclid's famous work "The Elements" is called Euclidean geometry. The salient feature of this geometry is that the fifth postulate, the parallel postulate, holds. It follows from this that through any point in the plane there is precisely one line through that point parallel to any given direction, that all angles in a triangle sum to precisely 180°, and that the ratio of the circumference of any circle to its diameter is always the same value π. Two-dimensional Euclidean geometry is called plane geometry, and the three-dimensional Euclidean geometry is called solid geometry. In 1899 German mathematician David Hilbert (1862–1943) proved that the theory of Euclidean geometry is free from contradiction.

Euclid of Alexandria (ca. 300–260 B.C.E.) began his famous 13-volume piece "The Elements" with 23 definitions ("a point is that which has no part" and "a line is that which has no breadth") followed by 10 axioms divided into two types: five common notions and five postulates.

His common notions were:

1. *Things that are equal to the same thing are equal to one another.*
2. *If equal things are added to equals, then the wholes are equal.*
3. *If equal things are subtracted from equals, then the remainders are equal.*
4. *Things that coincide with one another are equal to one another.*
5. *The whole is greater than the part.*

Euclid's postulates were:

1. *A straight line can be drawn to join any two points.*
2. *Any straight line segment can be extended to a straight line of any length.*
3. *Given any straight line segment, it is possible to draw a circle with centre one endpoint and with the straight line segment as the radius.*
4. *All right angles are equal to one another.*
5. *If two straight lines emanating from the endpoints of a given line segment have interior angles on one given side of the line segment summing to less than two right*

angles, then the two lines, if extended, meet to form a triangle on that side of the line segment.

It is worth noting that Euclid deliberately avoided any direct mention of the notion of infinity. His wording of the second postulate, for instance, avoids the need to state that straight lines can be extended indefinitely, and his fifth postulate, also known as the parallel postulate, avoids direct mention of parallel lines, that is, lines that never meet when extended indefinitely.

From these basic assumptions Euclid deduced, by pure logical reasoning, 465 statements of truth (theorems) about geometric figures. The systematic approach he followed and the rigour of reasoning he introduced was hailed as a great intellectual achievement. His model of mathematical exploration became the standard for all mathematical research for the next 2,000 years.

Euclid's fifth postulate was always regarded with suspicion. It was never viewed as simple and as self-evident as his remaining four postulates, and Euclid himself did his utmost to avoid using it in his work. (Euclid did not invoke the fifth postulate until his 29th proposition.) Over the centuries scholars came to believe that the fifth postulate could be logically deduced from the remaining four postulates and therefore did not need to be listed as an axiom. Many people proposed proofs for it, including the 5th-century Greek philosopher Proclus, who is noted for his historical account of Greek geometry. Unfortunately, his proof was flawed, as were the proofs proposed by Arab scholars of the 8th and 9th centuries, and by Western scholars of the Renaissance.

In 1733 Italian teacher and scholar Girolamo Saccheri (1667–1733) believed that because Euclid's axioms model the real world, which he thought to be consistent, they cannot lead to a contradiction. If the first four postulates do indeed imply that the fifth postulate is also true, then assuming the four postulates together with the negation of the fifth postulate should lead to a logical inconsistency. Unfortunately, in following this tact, Saccheri never came across a contradiction.

In 1795 Scottish mathematician and physicist John Playfair (1748–1819) proposed an alternative formulation of the famous fifth postulate (today known as Playfair's axiom). It states:

In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.

This version of the axiom is considerably easier to handle, and its negation is easier to envision. In an attempt to follow Saccheri's approach, Russian mathematician Nikolai Ivanovich Lobachevsky (1792–1856) and Hungarian mathematician János Bolyai (1802–1860), independently came to the same surprising conclusion: the first four of Euclid's postulates together with the negation of Playfair's version of the fifth postulate will not lead to a contradiction. This established, once and for all, that the fifth postulate is an independent axiom and cannot be deduced from the remaining four postulates. More important, by exploring the geometries that result in assuming that the fifth postulate does not hold, scholars were led to the discovery of non-Euclidean geometry.

In the late 1800s the German mathematician David Hilbert (1862–1943) noted that, despite its rigour, Euclid's work contained many hidden assumptions. He also realized, despite Euclid's attempts to describe them, that the notions of "point," "line," and "plane" cannot be properly defined and must remain as undefined terms in any theory of geometry. In his 1899 work "Foundations of Geometry" Hilbert refined and expanded Euclid's postulates into a list of 28 basic assumptions that define all that is needed in a complete account of Euclid's geometry. His axioms are today referred to as Hilbert's axioms.

(from Elementary Geometry for College Students)